DS 574 Algorithmic Mechanism Design **Lecture #13**
Fall 2023 Prof. Kira Goldner

Prophet Inequalities

You're at a casino gambling, and are offered to play the following game. Items will arrive one-by-one. As an item arrives, you see its value. You may only take a single item, and once you take an item, the game ends. A priori, you know the *distribution* of each item. At some point there will be a red item with the red distribution of values, and at some point there will be a blue item with the blue distribution of values, and so forth. However, you do not know the order of items (it is adversarial), and you do not know the exact values of the items (they are drawn from their specific distributions). Your goal is to come up with an algorithm that competes with the *prophet* who is all knowing, so knows the realization of values and the arrival order.

That is, n items will arrive in adversarial order. Item i (which is a label, not necessarily the order) has value v_i drawn from known distribution F_i. Your goal is to determine an algorithm ALG such that the value you get from gambling competes with the prophet who always gets $\max_i v_i$. However, your competition is over the randomness of the values that are drawn, so you only have to compete with $\text{OPT} = \mathbb{E}_{\mathbf{v}}[\max_i v_i]$.

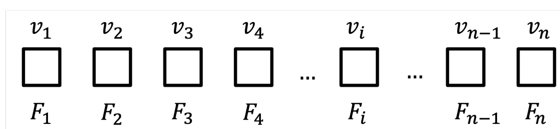

Figure 1: The prophet inequality problem.

To summarize:

- Goal: Pick one item; maximize its value.
- Gambler knows distribution for each item.
- Order is adversarial.
- Inspect each item online (see v_i) and irrevocably decide whether to take or pass forever.
- Compete with $\text{OPT} = \mathbb{E}_{\mathbf{v}}[\max_i v_i]$.

The Prophet Inequality problem was posed by Samuel-Cahn '84 [9], with the original solution and analysis that we'll see by Krengel Sucheston '78 [6] and Garling. It was brought to Algorithmic Mechanism Design by Hajiaghayi Kleinberg Sandholm '07 [3], and a new analysis for this case was developed by Kleinberg Weinberg '12 [4, 5].

Prove the following.

Theorem 1. *There is a threshold algorithm* ALG *such that when the gambler takes an item if and only if its value is above T,* $\text{ALG} \geq \frac{1}{2}\text{OPT}$.

Determine what threshold T to use and prove this statement using the following steps:

1. Divide what the algorithm yields from an item (in expectation) into exactly the threshold and the surplus above the threshold.

2. Lower bound your surplus term.

3. Set your threshold in order to combine like-terms and have OPT pop out.

Note: Can you find two different thresholds that give this same approximation?

Proof. We consider two different ways to set the threshold, starting a proof of what our algorithm obtains using the framework above. Let p denote the probability that *some* (at least one) $v_i \geq T$ for $i \in [n]$.

We will set the threshold T such that either (1) $T = \frac{1}{2}\mathbb{E}[\max_i v_i]$, or (2) such that $p = \frac{1}{2}$.

$$\begin{aligned}
\text{ALG} &= \Pr[\text{any above}]T + \sum_i \Pr[\text{all } j < i \text{ below}] \cdot \mathbb{E}[(v_i - T)^+] \\
&\geq pT + (1-p)\sum_i \mathbb{E}[(v_i - T)^+] \\
&\geq pT + (1-p)\mathbb{E}[\sum_i (v_i - T)^+] \\
&\geq pT + (1-p)(\mathbb{E}[\max_i v_i] - T) \qquad (*)
\end{aligned}$$

$$\text{using (1)} \qquad \begin{aligned}
(*) &\geq p\left(\frac{1}{2}\mathbb{E}[\max_i v_i]\right) + (1-p)\left(\frac{1}{2}\mathbb{E}[\max_i v_i]\right) \\
&= \frac{1}{2}\text{OPT}.
\end{aligned}$$

$$\text{using (2)} \qquad \begin{aligned}
(*) &= \frac{1}{2}T + \frac{1}{2}\left(\mathbb{E}[\max_i v_i] - T\right) \\
&= \frac{1}{2}\text{OPT}.
\end{aligned}$$

Hence a threshold algorithm set using (1) or (2) produces a $\frac{1}{2}$-approximation to the prophet (a $\frac{1}{2}$-competitive ratio). □

We reinterpret our problem as follows: we have a single item to sell, and n buyers arrive online with their values for the item drawn from distributions. We price the item at T. Observe that the first term above is the *expected revenue* earned from selling the item to the buyers at T, and the second term is the *expected utility*, or surplus. Welfare is equal to revenue + utility.

Exercise: You could see this as a mechanism for a buyer to maximize social welfare. Could you design a mechanism to maximize revenue using the prophet inequality?

[Hint: Use virtual values.]

See Roughgarden Twenty Lectures (364A) Lecture 6 Section 3 for a formal treatment on how to do this.

The Multidimensional Extension [1, 2]

Imagine now that a seller has m goods to sell, and n buyers will arrive online one at a time. They have a combinatorial value function $v_i : 2^m \to \mathbb{R}_{\geq 0}$ and each function v_i is drawn from a known prior distribution. Our goal is to set prices such that the welfare of the allocation is maximized.

That is, if we determine the allocation $\mathbf{x}$ such that x_i is the allocation to bidder i, we wish to set a pricing function $p : 2^m \to \mathbb{R}_{\geq 0}$ such that under online arrival, the buyer's allocations maximize $\sum_i v_i(x_i)$.

We will use the notation $\mathbf{x}$ to denote a partial allocation rule, for instance, the allocation of items after arrival so far. We will use $\mathbf{x}'$ to denote a second allocation rule that, combined with the partial allocation rule of $\mathbf{x}$, is still feasible. That is, in this online context, $\mathbf{x}'$ is a feasible allocation of the remaining items not allocated by $\mathbf{x}$.

We will use the notation $\text{OPT}(\mathbf{v} \mid \mathbf{x})$ to denote the optimal welfare of all allocations that are feasible with $\mathbf{x}$, i.e., the value remaining (not including the value from $\mathbf{x}$) when $\mathbf{x}$ is already committed to. Formally,

$$\text{OPT}(\mathbf{v} \mid \mathbf{x}) = \max_{\mathbf{x}' : \mathbf{x} \cup \mathbf{x}' \text{ feas.}} v_i(x_i').$$

Definition 1. A pricing rule p is (α, β)-*balanced* with respect to valuation profile $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ if, for all feasible allocations $\mathbf{x}$ and $\mathbf{x}'$ such that $\mathbf{x} \cup \mathbf{x}'$ is feasible:

1. $\sum_i p(x_i) \geq \frac{1}{\alpha} \cdot \underbrace{(\text{OPT}(\mathbf{v}) - \text{OPT}(\mathbf{v} \mid \mathbf{x}))}_{\text{value lost due to allocating } \mathbf{x}}$
2. $\sum_i p(x_i') \leq \beta \cdot \underbrace{\text{OPT}(\mathbf{v} \mid \mathbf{x})}_{\text{value remaining after allocating } \mathbf{x}}$

Prices are *weakly balanced* if the second condition is relaxed to $\text{OPT}(\mathbf{v})$.

We can think of the quantity on the right hand side of (1) as the externality with respect to already allocation $\mathbf{x}$.

Theorem 2. *If a pricing rule p is (α, β)-balanced with respect to valuations $\mathbf{v}$, then posting prices $\delta \cdot p$ guarantees value at least $\frac{1}{\alpha\beta+1} \cdot \text{OPT}(\mathbf{v})$ for $\delta = \frac{\alpha}{\alpha\beta+1}$.*

Sanity check: In the single-item case, a price $p = \max_i v_i$ is $(1, 1)$-balanced. Which implies?

A 1/2-approximation of welfare by setting $\frac{1}{2} \max_i v_i$ as the price.

Proof. Let $\mathbf{x}$ be the allocation sold, and let $\mathbf{x}'$ be the allocation achieving $\textsc{opt}(\mathbf{v} \mid \mathbf{x})$. Then we get that

$$\textsc{Revenue} = \sum_i \delta \cdot p(x_i) \geq \frac{\delta}{\alpha}(\textsc{opt}(\mathbf{v}) - \textsc{opt}(\mathbf{v} \mid \mathbf{x}))$$

$$\textsc{Utility} \geq \sum_i (v_i(x_i') - \delta p(x_i')) \geq (1 - \delta\beta)\textsc{opt}(\mathbf{v} \mid \mathbf{x})$$

Clearly

$$\textsc{Welfare}(\mathbf{x} \cup \mathbf{x}') \geq \textsc{Revenue}(\mathbf{x}) + \textsc{Utility}(\mathbf{x}').$$

Setting δ so that $\frac{\delta}{\alpha} = (1 - \delta\beta)$ concludes the proof. □

Extension to unknown values: For (α, β)-balanced p^v, let $p = \mathbb{E}_{v \sim F}[p^v]$. Then posting prices δp guarantees expected value $\frac{1}{\alpha\beta+1} \cdot \mathbb{E}[\textsc{opt}(\mathbf{v})]$. (For weakly balanced, it's $\frac{1}{4\alpha\beta}$.)

Construction of Balanced Prices: If any individual's allocation is bounded at k items, and if $\mathbf{x}^*$ is the optimal allocation, then we set

$$p_j = \begin{cases} \frac{1}{|x_i^*|} v_i(x_i^*) & j \in x_i^* \\ 0 & \text{otherwise} \end{cases}.$$

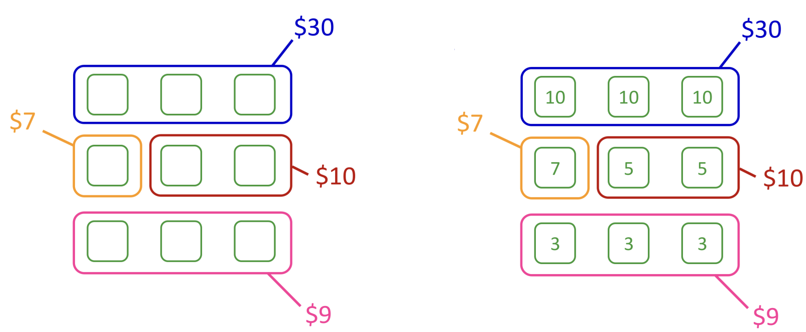

Figure 2: Construction of balanced prices. Figure from Brendan Lucier.

Claim 1. These are weakly $(k, 1)$-balanced.

Proof. Let $S(\mathbf{x})$ denote the items allocated by $\mathbf{x}$, that is, $S(\mathbf{x}) = \bigcup_i x_i$. Let $M(j)$ denote the index of the bidder who owns item j in allocation $\mathbf{x}^*$; that is, $j \in x_i^* \iff i = M(j)$. For condition (1), we see that

$$\begin{aligned}
\forall \mathbf{x}, \quad \sum_i p(x_i) &\geq \sum_{j \in S(\mathbf{x})} \frac{1}{|x^*_{M(j)}|} v_{M(j)}(x^*_{M(j)}) \\
&\geq \frac{1}{k} \sum_{j \in S(\mathbf{x})} v_{M(j)}(x^*_{M(j)}) + \frac{1}{k} \sum_{j \notin S(\mathbf{x})} v_{M(j)}(x^*_{M(j)}) - \frac{1}{k} \sum_{j \notin S(\mathbf{x})} v_{M(j)}(x^*_{M(j)}) \\
&= \frac{1}{k} \cdot (\textsc{opt}(\mathbf{v}) - \textsc{opt}(\mathbf{v} \mid \mathbf{x}))
\end{aligned}$$

That is, for any $\mathbf{x}$, the sum of prices of $\mathbf{x}$ is at least $1/k$ of the value of allocations in OPT that intersect $\mathbf{x}$.

For condition (2), we see

$$\forall \mathbf{x}', \quad \sum_i p(x'_i) \leq \sum_{j \notin S(\mathbf{x})} \frac{1}{|x^*_{M(j)}|} v_{M(j)}(x^*_{M(j)}) \leq 1 \cdot \text{OPT}(\mathbf{v}).$$

Observe that the right hand side does not hold for $\text{OPT}(\mathbf{v} \mid \mathbf{x})$ because item j's price is the *average* value of the bundle that item j is in in the optimal allocation. However, the value of item j itself may be far higher than the average of the bundle, so this quantity may not be bounded by $\text{OPT}(\mathbf{v} \mid \mathbf{x})$.

That is, after removing $\mathbf{x}$, the total price of the items left over is at most the total price of ALL items, which is $\text{OPT}(\mathbf{v})$.

Our conclusion is then that there exist items prices that guarantee a $4k$-approx. to the optimal expected welfare. □

Additional Resources

I'm not aware of any textbooks on the subject, but here is a list of resources on prophet inequalities and the breadth of work in more recent research:

- 2017 Survey "An Economic View of Prophet Inequalities" by Brendan Lucier [7]: https://sigecom.org/exchanges/volume_16/1/LUCIER.pdf
- 2016 Simons Bootcamp Talks by Matt Weinberg
  - Part I: https://www.youtube.com/watch?v=NwF4Xr0-6Rc
  - Part II: https://www.youtube.com/watch?v=E19TWolvn8I
- EC 2021 Tutorial on Prophet Inequalities by Michal Feldman, Thomas Kesselheim, and Sahil Singla
  - Website (slides and reading list): http://www.thomas-kesselheim.de/tutorial-prophet-inequalities/
  - Part 1: https://www.youtube.com/watch?v=qbHd0g9RkCg
  - Part 2: https://www.youtube.com/watch?v=l20KP5IIgcQ
  - Part 3: https://www.youtube.com/watch?v=lyOUcYfNEiA

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References

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