DS 574 Algorithmic Mechanism Design **Lecture #2**
Fall 2023 Prof. Kira Goldner

Myerson's Lemma

We now come to two important definitions. Both articulate a property of allocation rules.

Definition 1 (Implementable Allocation Rule)**.** An allocation rule $\mathbf{x}$ is *implementable* if there is a payment rule $\mathbf{p}$ such the sealed-bid auction $(\mathbf{x}, \mathbf{p})$ is DSIC.

Definition 2 (Monotone Allocation Rule)**.** An allocation rule x for a single-parameter environment is *monotone* if for every bidder i and bids $\mathbf{b}_{-i}$ by the other bidders, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is nondecreasing in its bid z.

That is, in a monotone allocation rule, bidding higher can only get you more stuff.

For example, the single-item auction allocation rule that awards the good to the highest bidder is monotone: if you're the winner and you raise your bid (keeping other bids constant), you continue to win. By contrast, awarding the good to the second-highest bidder is a non-monotone allocation rule: if you're the winner and raise your bid high enough, you lose.

We state Myerson's Lemma in three parts; each is conceptually interesting and will be useful in later applications.

Theorem 1 (Myerson's Lemma Myerson [1981])**.** *Fix a single-parameter environment.*

(a) An allocation rule $\mathbf{x}$ *is implementable if and only if it is monotone.*

(b) If $\mathbf{x}$ *is monotone, then there is a unique payment rule such that the sealed-bid mechanism* $(\mathbf{x}, \mathbf{p})$ *is DSIC [assuming the normalization that* $b_i = 0$ *implies* $p_i(\mathbf{b}) = 0$*].*

(c) The payment rule in (b) is given by an explicit formula:

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i}) dz.$$

Myerson's Lemma is the foundation on which we'll build most of our mechanism design theory. Let's review what it is saying.

Part (a): Finding an allocation rule that can be made DSIC (is implementable, Definition 1) seems confusing, but is actually equivalent to and just as easy as checking if the allocation is monotone (Definition 2).

Part (b): If an allocation rule *is* implementable (can be made to be DSIC), then there's no ambiguity in what the payment rule should be.

Part (c): There's a simple and explicit formula for this!

Proof of Myerson's Lemma (Theorem 1). As shorthand, write $x(z)$ and $p(z)$ for the allocation $x_i(z, \mathbf{b}_{-i})$ and payment $p_i(z, \mathbf{b}_{-i})$ of i when it bids z, respectively.

Suppose $(\mathbf{x}, \mathbf{p})$ is DSIC, and consider any $0 \le y < z$. Because bidder i might well have private valuation z and can submit the false bid y if it wants, DSIC demands that

$$\underbrace{z \cdot x(z) - p(z)}_{\text{utility of bidding } z \text{ given value } z} \quad \ge \quad \underbrace{z \cdot x(y) - p(y)}_{\text{utility of bidding } y \text{ given value } z} \tag{1}$$

Similarly, since bidder i might well have the private valuation y and could submit the false bid z, $(\mathbf{x}, \mathbf{p})$ must satisfy

$$\underbrace{y \cdot x(y) - p(y)}_{\text{utility of bidding } y \text{ given value } y} \quad \ge \quad \underbrace{y \cdot x(z) - p(z)}_{\text{utility of bidding } z \text{ given value } y} \tag{2}$$

Rearranging inequalities (1) and (2) yields the following sandwich, bounding $p(y) - p(z)$ from below and above:

$$y \cdot [x(z) - x(y)] \le p(z) - p(y) \le z \cdot [x(z) - x(y)] \tag{3}$$

From here, we can conclude:

- $\mathbf{x}$ must be monotone.
- $p'(z) = z \cdot x'(z)$.

Why? First, if x is not monotone, the inequalities in (3) would be violated. Second, assuming x is differentiable, by dividing (3) by $z - y$ and taking the limit as $y \to z$, we obtain $p'(z) = z \cdot x'(z)$. Even for non-differentiable x, we obtain a similar equation in terms of the change in the allocation at z.

Assuming that $p(0) = 0$ then gives the payment identity

$$p_i(b_i, \mathbf{b}_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z, \mathbf{b}_{-i}) dz$$

or alternatively, after integration by parts,

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i}) dz \tag{4}$$

for every bidder i, bid b_i, and bids $\mathbf{b}_{-i}$ by the others.

Equation (3) tells us that this is the only payment rule that could possibly be DSIC. But does it in fact satisfy DSIC when x is monotone?

Bidder i's utility will then be

$$u_i(b_i, \mathbf{b}_{-i}) = v_i \cdot x_i(b_i, \mathbf{b}_{-i}) - p_i(b_i, \mathbf{b}_{-i}),$$

or with the payment identity,

$$u_i(b_i, \mathbf{b}_{-i}) = (v_i - b_i) \cdot x_i(b_i, \mathbf{b}_{-i}) + \int_0^{b_i} x_i(z, \mathbf{b}_{-i}) dz$$

which for monotone $\mathbf{x}$ is maximized when $b_i = v_i$, independent of $\mathbf{b}_{-i}$, as desired. □

Single-Parameter Environments

All of our definitions and Myerson's Lemma actually apply to a more general setting which we call *single-parameter environments*. The main idea here is that each bidder i only has a single piece of private information, like their value v_i, that needs to be elicited in order to run the mechanism. Here are some other examples of non-single-item yet single-parameter environments.

- **Single-item:** A seller has a single item to sell. The set of feasible outcomes X satisfy $\sum_{i=1}^{n} x_i \leq 1$ and $x_i \in \{0, 1\}$.
- **k identical items:** A seller has k identical items to sell and each buyer gets at most one. The set of feasible outcomes X satisfy $\sum_{i=1}^{n} x_i \leq k$ and $x_i \in \{0, 1\}$.
- **Sponsored search:** There are k advertising slots, each with click-through-rate α_j. A buyer i gets value $v_i \cdot \alpha_j$ from winning the jth slot. The set of feasible outcomes X satisfy $\sum_{i=1}^{n} x_i \leq \sum_{j=1}^{k} \alpha_j$ and $x_i \in \{\alpha_j\}_{j=1}^{k} \cup \{0\}$ where $x_i = \alpha_j$ if bidder i is assigned the jth slot.

Exercise (optional): Graph an allocation rule as a function of a single-bidder (hold $\mathbf{b}_{-i}$ fixed) with value on the x-axis and allocation on the y-axis. Show that for a DSIC auction, Myerson's Lemma implies that the payment is the area to the left of the allocation curve, and a bidder's utility is the area under the allocation curve.

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References

Roger B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981. URL `http://dx.doi.org/10.1287/moor.6.1.58`.

Tim Roughgarden. *Twenty lectures on algorithmic game theory*. Cambridge University Press, 2016.