DS 574 Algorithmic Mechanism Design	**Lecture #5**
Fall 2022	Prof. Kira Goldner

Single-Parameter Optimal Revenue (continued)

Virtual Welfare Recap

- Maximize welfare ($\sum_i v_i x_i$): Always give the bidder the item, always give it away for free!
- Maximize revenue: Post a price that maximizes $\textsc{Rev} = \max_r r \cdot [1 - F(r)]$.

Using only the revelation principle and the payment identity $p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz$, we proved the following:

$$\textsc{Revenue} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i \varphi_i(v_i) x_i(\mathbf{v})] = \textsc{Virtual Welfare}$$

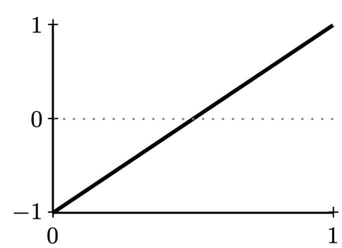
where

$$\varphi_i(v_i) = v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}.$$

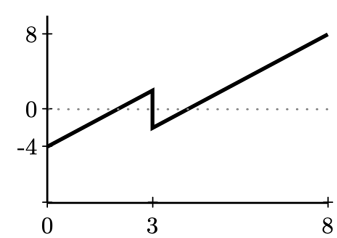
Then similarly to welfare, just give the item to the bidder with the highest (non-negative) *virtual* value! But this doesn't work when $\varphi(\cdot)$ isn't monotone, because then $x(\cdot)$ wouldn't be.

Definition 1. A distribution F is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

Claim 1. A virtual welfare maximizing allocation x is monotone if and only if the virtual value functions are regular.



(a) Uniform agent virtual value.



(b) Bimodal agent virtual value.

Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

Quantile Space and Ironing

In *value space*:

- an agent has value v.
- the fraction of the distribution with value above v is $1 - F(v)$.
- the revenue from posting a "take-it-or-leave-it" price of v is $v[1 - F(v)]$.

In *quantile space*: $q = 1 - F(v)$.

- an agent has value v.
- the fraction of the distribution with value above v is $q(v) = 1 - F(q)$.
- the revenue from posting a "take-it-or-leave-it" price of $v(q) = F^{-1}(1 - v)$ is $v(q) \cdot q$.

Example: Consider a distribution that is $U[\$0, \$10]$. Then the quantile 0.1 corresponds to \$9, where 10% of the population might have a higher value. We let $v(q)$ denote the corresponding value, so $v(0.1)$ is \$9.

Definition 2. The *quantile* of a single-dimensional agent with value $v \sim F$ is the measure with respect to F of stronger values, i.e., $q = 1 - F(v)$; the inverse demand curve maps an agent's quantile to her value, i.e., $v(q) = F^{-1}(1 - q)$.

Quantile Distribution: Quantiles are particularly useful because we can draw an agent from any distribution by drawing a quantile $q \sim U[0, 1]$. That is, for any $\hat{q}$ and any distribution F, $\Pr_F[q \leq \hat{q}] = \hat{q}$. In English: the probability that an agent has a value in the top 0.3 of the distribution is 0.3.

Note: For everything we do today, we *could* stay in value space, (and sometimes we'll compare), but we'd have to normalize by the distribution using $f(v)$, which makes everything a bit messier and a bit trickier.

Example: For the example of a uniform agent where $F(z) = z$, the inverse demand curve is $v(q) = 1 - q$.

For an allocation rule $x(\cdot)$ in value space, we define an allocation rule in *quantile space* $y(\cdot)$:

$$y(q) = x(v(q)).$$

As $x(\cdot)$ is monotone weakly increasing, then $y(\cdot)$ is monotone *weakly decreasing*.

Definition 3. The *revenue curve* of a single-dimensional agent specified by $R(v) = v \cdot [1 - F(v)]$.

Note: This is *only* the revenue that can be achieved by posting a single take-it-or-leave-it price. This does not capture the expected revenue of any given mechanism.

Definition 4. The *revenue curve* of a single-dimensional linear agent specified by inverse demand curve $v(\cdot)$ is $P(q) = q \cdot v(q)$ for any $q \in [0, 1]$.

Assuming the lower-end of the support of F is 0 and the upper end is some finite $v_{\max}$, then $P(0) = 0$ and $P(1) = 0$.

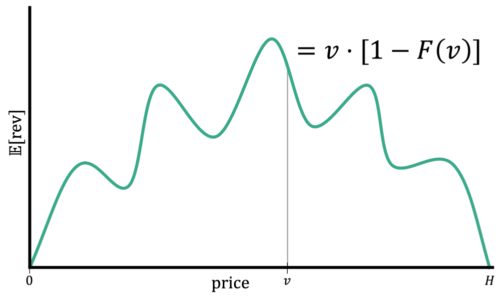


Figure 2: A revenue curve in *value space*.

Claim 2. Any allocation rule $y(\cdot)$ can be expressed as a distribution of posted prices.

Proof. Given the allocation rule $y(\cdot)$, consider the distribution $G^y(z) := 1 - y(z)$. We show that the mechanism that randomly draws a quantile $\hat{q} \sim G^y$ from the distribution G^y and posts the price $v(\hat{q})$ is equivalent.

For a random price $v(\hat{q})$ and fixed quantile q, then

$$\Pr_{\hat{q}\sim G^y}[v(\hat{q}) < v(q)] = \Pr_{\hat{q}\sim G^y}[\hat{q} > q] = 1 - G^y(q) = y(q).$$

□

Claim 3. Any DSIC allocation rule $x(\cdot)$ can be expressed as a distribution of posted prices.

See Figure for an example. In general, the PDF of the distribution of randomized prices is $x'(v)$ for a price of v to achieve an allocation rule of v..

Claim 4. A distribution F is regular if and only if its corresponding revenue curve is concave.

Observe that $P'(q) = \varphi(v(q))$:

$$P'(q) = \frac{d}{dq}\left(q \cdot v(q)\right) = v(q) + qv'(q) = v - \frac{1 - F(v)}{f(v)} = \varphi(v(q)).$$

Thus $\Phi(q) = \int_0^q \varphi(\hat{q})\, d\hat{q} = P(q)$.

To summarize: a distribution F is regular if and only if:

- its corresponding revenue curve *in quantile space* is concave.
- $\varphi(q)$ is strictly increasing.
- $f(v)\varphi(v)$ is strictly increasing. (Why?)

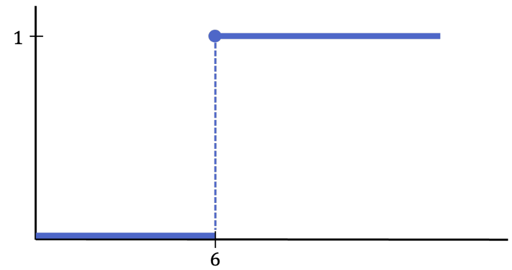
Claim 5. A distribution F is regular if and only if its corresponding revenue curve is concave.

Observe that $P'(q) = \varphi(v(q))$:

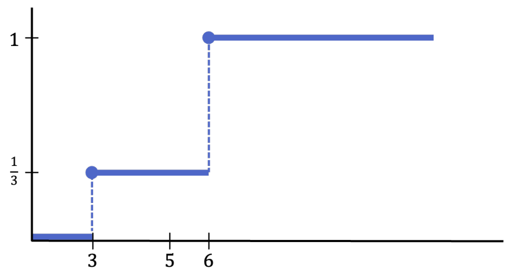
$$P'(q) = \frac{d}{dq}\left(q \cdot v(q)\right) = v(q) + qv'(q) = v - \frac{1 - F(v)}{f(v)} = \varphi(v(q)).$$

Thus $\Phi(q) = \int_0^q \varphi(\hat{q})\, d\hat{q} = P(q)$.

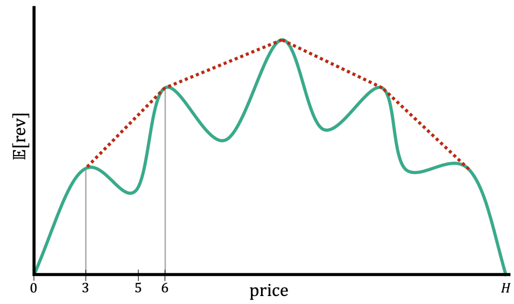
(a) Price of \$3.



(b) Price of \$6.



(c) Randomized price with expectation \$5.



(d) Ironed revenue curve.

Figure 3: (a) An allocation rule for a take-it-or-leave-it price of \$3. (b) An allocation rule for a take-it-or-leave-it price of \$6. (c) An allocation that can be written $x(v) = 0$ for $v < 3$, $x(v) = \frac{1}{3}$ for $v \in [3, 6)$, and $x(v) = 1$ for $v \geq 6$. Alternatively, a randomized take-it-or-leave-it price that is \$3 with probability $\frac{1}{3}$ and \$6 with probability $\frac{2}{3}$, that is, $\$5 = \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 6$ in expectation. (d) The revenue curve in value space, including ironed intervals where convex combinations of prices can attain higher revenue than deterministic prices.

Definition 5. The *ironing procedure* for (non-monotone) virtual value function φ (in quantile space) is:

(i) Define the cumulative virtual value function as $\Phi(\hat{q}) = \int_0^{\hat{q}} \varphi(q)\, dq$.

(ii) Define ironed cumulative virtual value function as $\bar{\Phi}(\cdot)$ as the concave hull of $\Phi(\cdot)$.

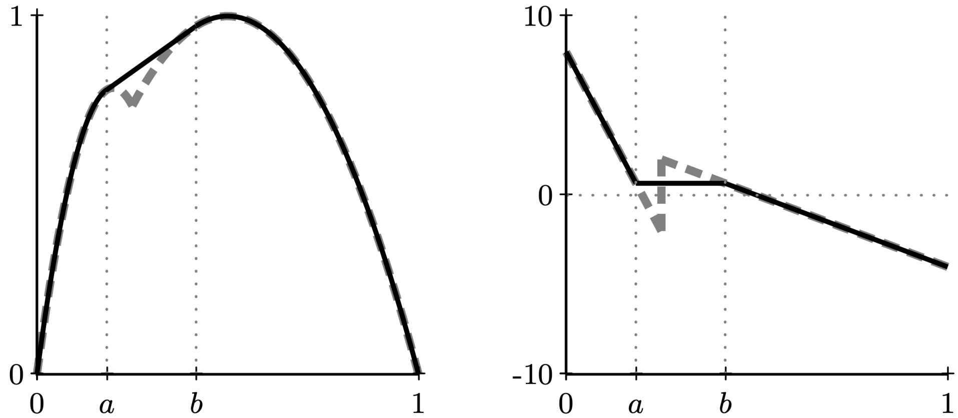
(iii) Define the ironed virtual value function as $\bar{\varphi}(q) = \frac{d}{dq}\bar{\Phi}(q) = \bar{\Phi}'(q)$.

Summary: Take the concave hull of the revenue curve in quantile space. Its derivative forms the ironed virtual values. (The derivatives of the original curve are the original virtual values.)

Theorem 1. *For any monotone allocation rule $y(\cdot)$ and any virtual value function $\varphi(\cdot)$, the expected virtual surplus of an agent is upper-bounded by her expected ironed virtual surplus, i.e.,*

$$\mathbb{E}[\varphi(q)y(q)] \leq \mathbb{E}[\bar{\varphi}(q)y(q)].$$

Furthermore, this inequality holds with equality if the allocation rule y satisfies $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$.



(a) Ironed cumulative virtual values. (b) Ironed virtual values.

Figure 4: The bimodal agent's (ironed) revenue curve and virtual values in quantile space.

Proof. Recall integration by parts:

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx.$$

By integration by parts for any virtual value function $\varphi(\cdot)$ and monotone allocation rule $y(\cdot)$,

$$\mathbb{E}[\varphi(q)y(q)] = \mathbb{E}[-y'(q)\Phi(q)].$$

Step by step, that is,

$$\begin{aligned}\mathbb{E}[\varphi(q)y(q)] &= \int_0^1 \varphi(q)y(q)\,dq && q \sim U[0,1]\\ &= \Phi(1)y(1) - \Phi(0)y(0) - \int_0^1 y'(q)\Phi(q)\,dq\\ &= 0 + \mathbb{E}[-y'(q)\Phi(q)].\end{aligned}$$

because $\Phi(1) = 1 \cdot v(1) = 0$ as $v(1) = 0$, and $\Phi(0) = 0 \cdot v(0) = 0$. Notice that the weakly decreasing monotonicity of the allocation rule $y(\cdot)$ implies the non-negativity of $-y'(q)$. With the left-hand side of equation as the expected virtual surplus, it is clear that a higher cumulative virtual value implies no lower expected virtual surplus. By definition of $\bar{\Phi}(\cdot)$ as the concave hull of $\Phi(\cdot)$, $\Phi(q) \leq \bar{\Phi}(q)$ and, therefore, for any monotone allocation rule, in expectation, the ironed virtual surplus is at least the virtual surplus, i.e., $\mathbb{E}[-y(q)\Phi(q)] \leq \mathbb{E}[-y(q)\bar{\Phi}(q)]$.

To see the equality under the assumption that $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$, rewrite the difference between the ironed virtual surplus and the virtual surplus via equation as,

$$\mathbb{E}[\bar{\varphi}(q)y(q)] - \mathbb{E}[\varphi(q)y(q)] = \mathbb{E}[-y'(q)(\bar{\Phi}(q) - \Phi(q))].$$

The assumption on y' implies the term inside the expectation on the right-hand side is zero $\forall q$. $\square$

Modifying this statement for value space:

Theorem 2. *For any monotone allocation rule $x(\cdot)$ and any virtual value function $\varphi(\cdot)$, the expected virtual welfare of an agent is upper-bounded by their expected ironed virtual welfare, i.e.,*

$$\mathbb{E}[\varphi(v)x(v)] \leq \mathbb{E}[\bar{\varphi}(v)x(v)].$$

Furthermore, this inequality holds with equality if the allocation rule x satisfies $x'(v) = 0$ for all v where $\bar{\Phi}(v) > \Phi(v)$.

Claim 6. The expected revenue on the ironed revenue curve is attainable with a DSIC mechanism.

Example: How would you obtain the ironed revenue at \$5 instead of just $R(5)$?

For $p \in [\underline{p}, \overline{p}]$ where $\overline{R}(p) > R(p)$, if $p = \alpha\underline{p} + (1-\alpha)\overline{p}$, we achieve $\overline{R}(p)$ by randomizing the prices $\underline{p}$ and $\overline{p}$ with probabilities α and $1-\alpha$ accordingly to yield $\alpha R(\underline{p}) + (1-\alpha)R(\overline{p})$ on the concave closure.

Note: Recall that the expected revenue of *any mechanism*, not just a posted price, can be expressed by its virtual welfare. (We have now shown that you could decompose it into a distribution of posted prices and thus express the revenue that way, too, actually.)

What's the final mechanism? Now that $\bar{\varphi}_i(\cdot)$ is monotone (for every i), we choose the $x(\cdot)$ that maximizes $\mathbb{E}_v[\sum_i \varphi_i(v)x_i(v)]$, which will thus be monotone. By Theorem 2, this is an *upper bound* on the optimal revenue.

For any ironed interval $[a, b]$, examine $\bar{\varphi}(v)$ for $v \in [a, b]$. $P(q(v))$ is a straight line (linear) there, so $\bar{\varphi}(q(v))$ will be constant.

What does this imply for ironed-virtual-welfare-maximizing allocation in $[a, b]$? It will be constant on $[a, b]$, and thus its derivative will be zero.

Hence ironed virtual welfare is equal to virtual welfare by Theorem 2, so maximizing one maximizes the other.

Multiple Bidders

Imagine we have three bidders competing in a revenue-optimal auction for a single item. They are as follows:

- Bidder 1 is uniform. $F_1(v) = \frac{v-1}{H-1}$ on $[1, H]$.
- Bidder 2 is exponential. $F_2(v) = 1 - e^{-v}$ for $v \in (1, \infty)$.
- Bidder 2 is exponential. $F_3(v) = 1 - e^{-2v}$ for $v \in (1, \infty)$.

What does the optimal mechanism look like?

First we calculate their virtual value functions.

- $f_1(v) = \frac{1}{H-1}$ for $v \in [1, H]$. $\varphi_1(v) = 2v - H$.
- $f_2(v) = e^{-v}$ for $v \in (1, \infty)$. $\varphi_2(v) = v - 1$.
- $f_3(v) = 2e^{-2v}$ for $v \in (1, \infty)$. $\varphi_3(v) = v - \frac{1}{2}$.

The bidders have personalized reserve prices (i.e., have positive virtual values with v_i above) $r_1 = \frac{H}{2}$, $r_2 = 1$, $r_3 = \frac{1}{2}$. Note that based on the support of F_2 and F_3 that bidder 2 and 3 are always above their reserve prices.

The optimal mechanism excludes bidder 1 if $v_1 < r_1 = \frac{H}{2}$, and otherwise allocates to the bidder with the largest virtual value $\varphi_i(v_i)$. If some $\varphi_j(v_j)$ is the second highest virtual value and exceeds its reserve price, then bidder i pays a price of $\varphi_i^{-1}(\varphi_j(v_j))$; otherwise, bidder i just pays r_i.

Definition 6. A *reserve price* r is a minimum price below which no buyer may be allocated the item. There may also be personalized reserve prices r_i where if $v_i < r_i$ then v_i will not be allocated to. Bidders above their reserves participate in the auction.

Acknowledgements

This lecture was developed in part using materials by Jason Hartline, and in particular, his book "Mechanism Design and Approximation" [1].

References

[1] Jason D. Hartline. Mechanism design and approximation. *Book draft. October*, 122, 2013.