DS 574 Algorithmic Mechanism Design	**Lecture #9 Worksheet**
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Linear Programming Duality

The Dual of a Linear Program

Every linear program has a *dual* linear program. We call the original linear program the *primal*. There are a bunch of amazing properties that come from LP duality.

Going back to our nutrition example, we want to find the dual linear program. A maximization problem's dual is a minimization problem. Here, we have a minimization problem, so the dual will be a maximization problem.

To take the dual: Label each primal constraint with a new dual variable. In our new linear program, each dual constraint will correspond to a primal variable. For the left-hand side, count up the appearances of this constraint's primal variable (e.g., x_1) in each of the primal constraints and multiply them by the dual variable for those constraints. That is, if x_1 appears 5 times ($5x_1$) in constraint for y_1, then add $5y_1$ to x_1's constraint. Don't forget to include its appearance in the primal's objective function, but this will be the right-hand side of the constraint. Finally, the dual objective function is given by the right-hand side coefficients and their correspondence to the dual variables via the constraints in the primal. (See below).

Primal:

$$\begin{aligned} \min \quad & 0.6y_1 + 0.35y_2 & \\ \text{subject to} \quad & 5y_1 + 7y_2 \geq 8 & \text{(starch)} \\ & 4y_1 + 2y_2 \geq 15 & \text{(proteins)} \\ & 2y_1 + 1y_2 \geq 3 & \text{(vitamins)} \\ & y_1, y_2 \geq 0 & \text{(non-negativity)} \end{aligned}$$

Dual:

Sometimes, the dual can even be interpreted as a related problem.

The following is the normal form for a maximization problem primal and its dual:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

For the above example:

$\mathbf{A} =$ $\qquad$ $\mathbf{b} =$ $\qquad$ $\mathbf{c} =$

Example 3: Unweighted Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges S such that no vertex is covered by more than one edge.

Decision variables: x_e indicating whether edge e is in the matching.

Primal Linear Program:

$$\begin{aligned} \max \quad & \sum_{e \in E} x_e & & \\ \text{subject to} \quad & \sum_{e : v \in e} x_e \leq 1 & \forall v \quad & \text{(vertex matched at most once)} \quad (y_v) \\ & x_e \geq 0 & \forall e \quad & \text{(non-negativity)} \end{aligned}$$

Taking the dual of the above primal, we get the following linear program:

What problem is this?

Conditions for Optimality

Weak Duality

Theorem 1. *If* $\mathbf{x}$ *is feasible in (P) and* $\mathbf{y}$ *is feasible in (D) then* $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Proof.

What is not trivial (or by definition) is *strong duality*, and in fact, it is so involved that we will not even prove the hard direction: that an optimal solution always exists.

Strong Duality

Theorem 2 (Strong Duality)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if* $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.

Proof. $(\Rightarrow)$ Skip.
$(\Leftarrow)$

Complementary Slackness

Primal (P):

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \sum_i a_{ji}x_i \leq b_j \quad \forall j \quad (y_j) \\ & x_i \geq 0 \quad \forall i \end{aligned}$$

Dual (D):

$$\begin{aligned} \min \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \sum_i a_{ij}y_i \geq c_i \quad \forall i \quad (x_i) \\ & y_j \geq 0 \quad \forall j \end{aligned}$$

Theorem 3 (Complementary Slackness)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if the following complementary slackness conditions (1) and (2) hold:*

Proof.

Maximizing Welfare in the Unit Demand Setting

Given n unit-demand bidders and m items, determine the allocation rule that maximizes welfare. Do this by formulating a linear program.

Determine your objective, decision variables, and constraints.

Formulate the dual.

Do you see an interpretation of this dual?

Separation Oracles

Fact 1 (Ellipsoid Algorithm)**.** Every linear program that admits a polynomial-time separation oracle can be solved in polynomial time.

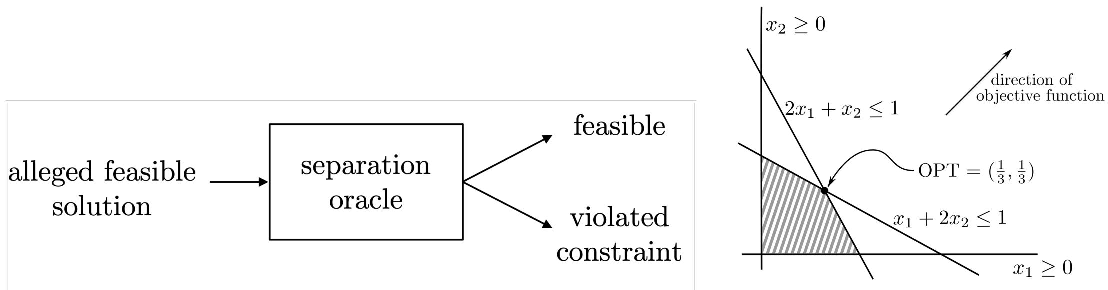

Figure 1: Left: A sketch of a separation oracle. For example, in the toy example on the right, on the alleged feasible solution $(\frac{1}{3}, \frac{1}{2})$, the separation oracle may return the violated constraint $x_1 + 2x_2 \leq 1$.

Consider a linear program such that:

a. There are n decision variables.

b. There are any number of constraints, for example, exponential in n. These constraints are not provided explicitly as input.

c. There is a polynomial-time *separation oracle* for the set of constraints. By "polynomial-time," we mean running time polynomial in n and the maximum number of bits of precision required. A separation oracle (Figure 1) is a subroutine that takes as input an alleged feasible solution to the LP, and either (i) correctly declares the solution to be feasible, or (ii) correctly declares the solution to be infeasible, and more strongly provides a proof of infeasibility in the form of a constraint that the proposed solution violates.

(The ellipsoid algorithm is not actually practical, but there are other algorithms that *are* often practically useful that rely on a separation oracle, such as cutting plane methods.)

Revenue Maximization

Consider the single-item revenue-maximization setting with n bidders. Formulate the LP to maximize revenue. Formulate its dual.