# Menu Complexity

## for the Space Between Single- and Multi-Dimensional Mechanism Design

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# Recap of Before the Break

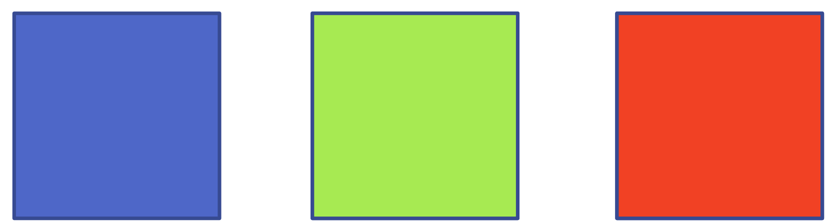
# 3 items for sale

**Goal:** Determine who gets what and who pays what

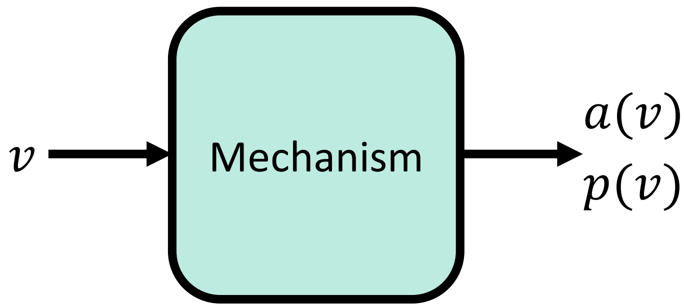
Identical:
- "single-dimensional"

All different:
- "multi-dimensional"
  - Combinatorial valuations
  - Additive Valuations
  - Independent valuations

Something in between?

# Taxation and Menus




Equivalent

Menu

$(a(v_1), p(v_1))$

$(a(v_2), p(v_2))$

$(a(v_3), p(v_3))$

Let buyer pick favorite

**Question:** What **size of menu** is needed to guarantee revenue?

Buyer with $\boldsymbol{v}$ picks own option

Incentive-compatibility (truthfulness): For all **w**, $u(v) > u(\boldsymbol{w} \mid v)$

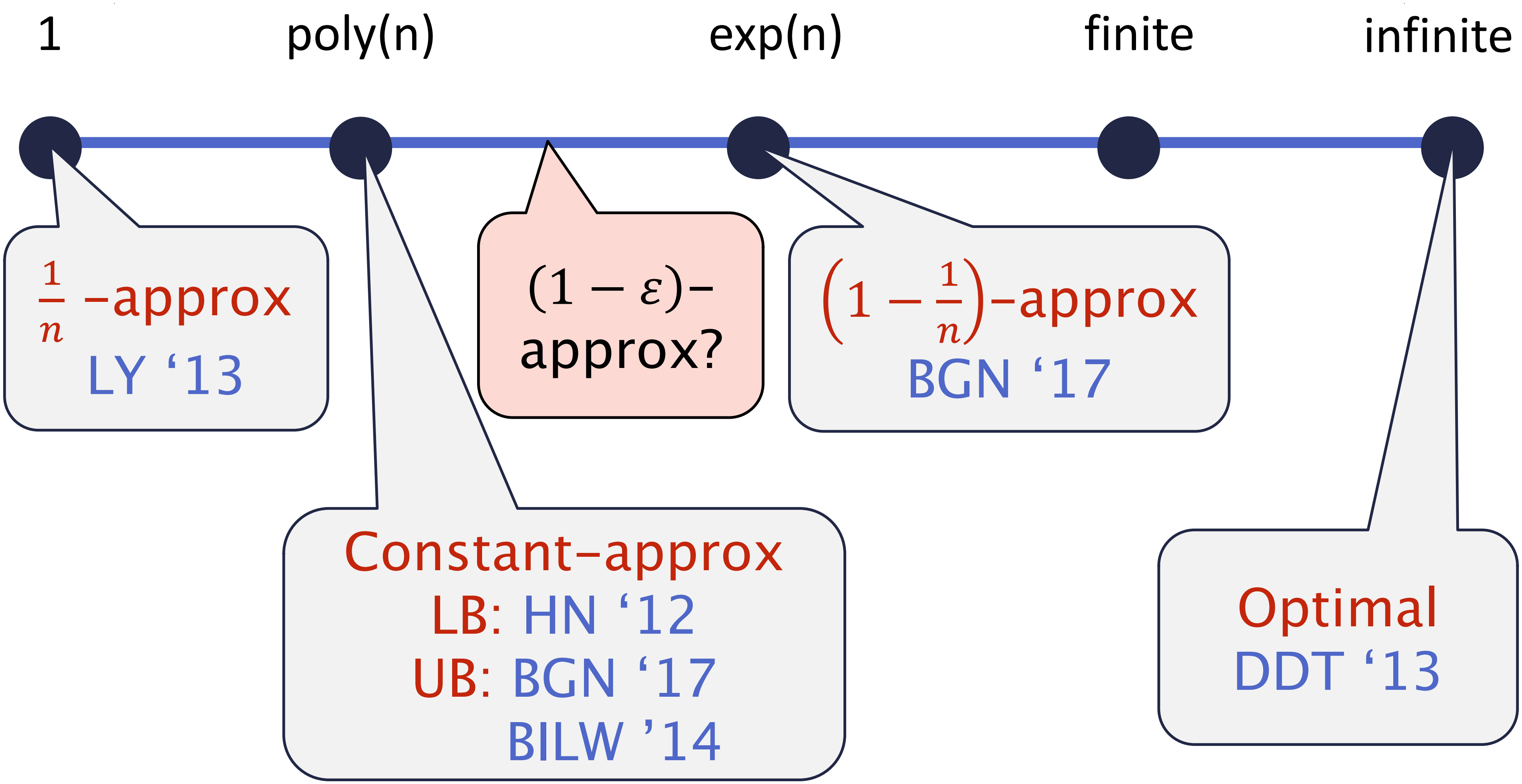
Restricting to IC (truthful) mechanisms is without loss.

# *n* items for sale

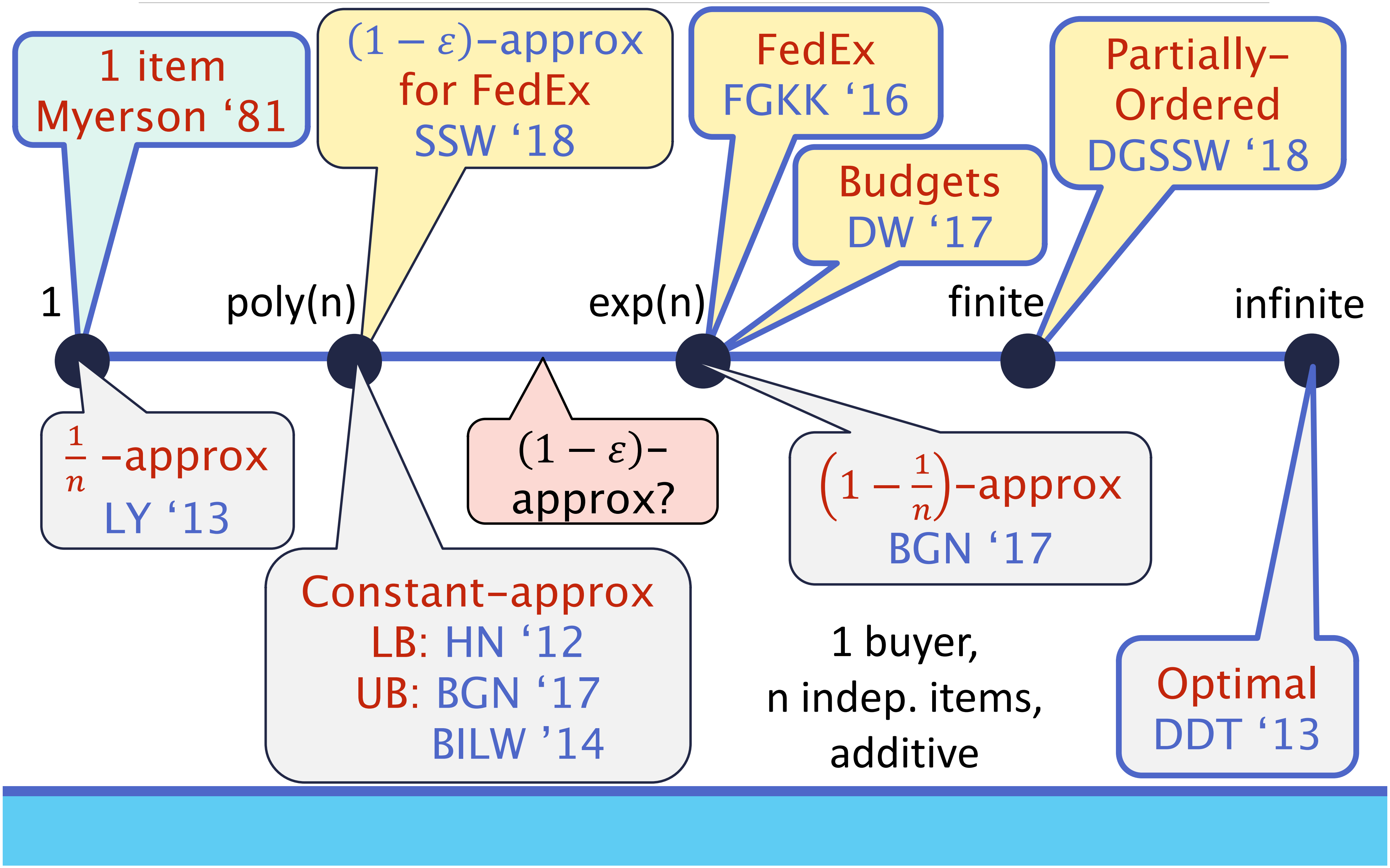
| | Optimal | Approximate |
|---|---|---|
| Identical:<br>◦ "single-dimensional" | menu size 1 | menu size 1 |
| All different:<br>◦ "multi-dimensional" | | |
| ◦ Combinatorial valuations | infinite menu size | infinite menu size |
| ◦ Additive Valuations | infinite menu size | infinite menu size |
| ◦ Independent valuations | infinite menu size | finite menu size |

**Something in between?**

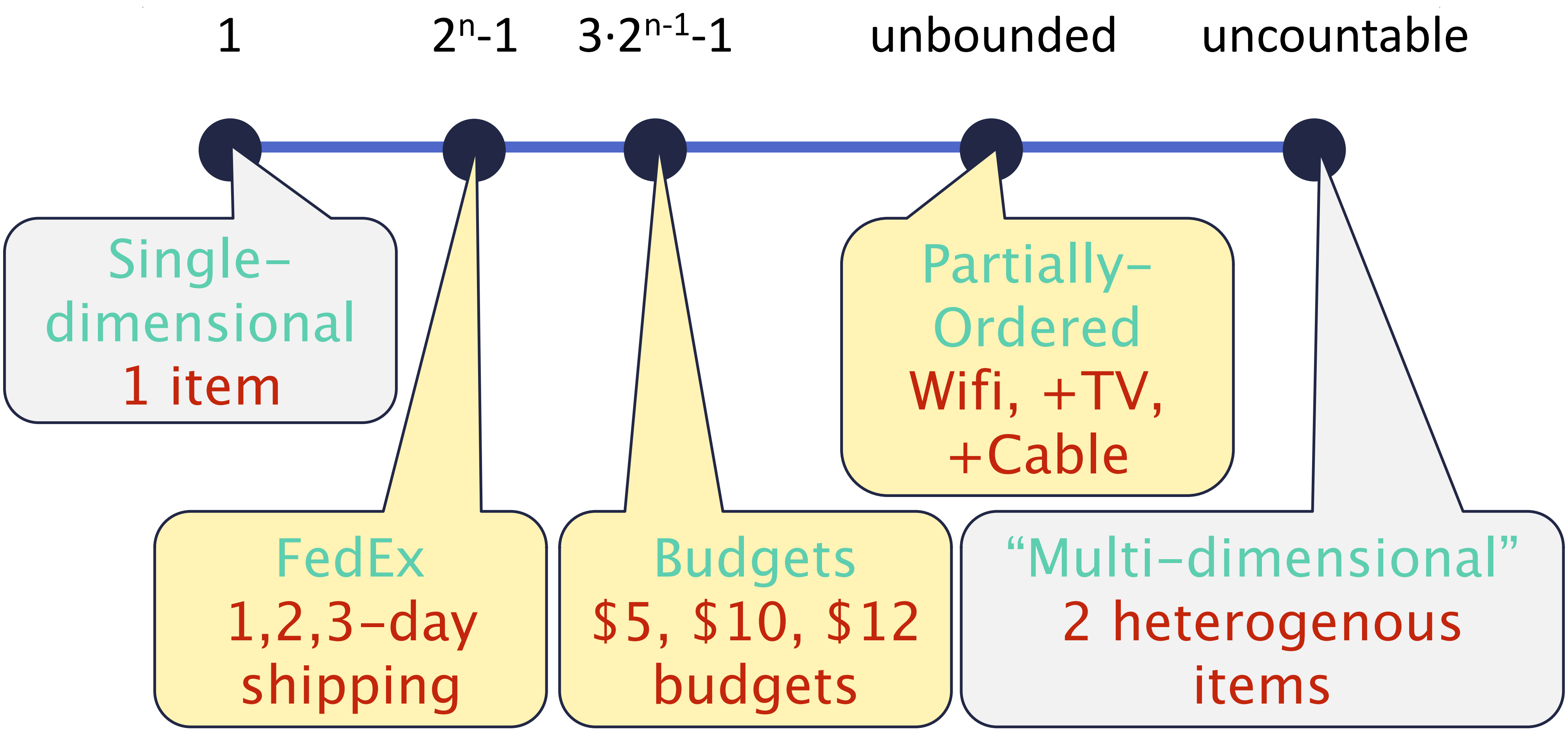
# Menu Complexity for Approximation:
## 1 buyer, additive over n independent items



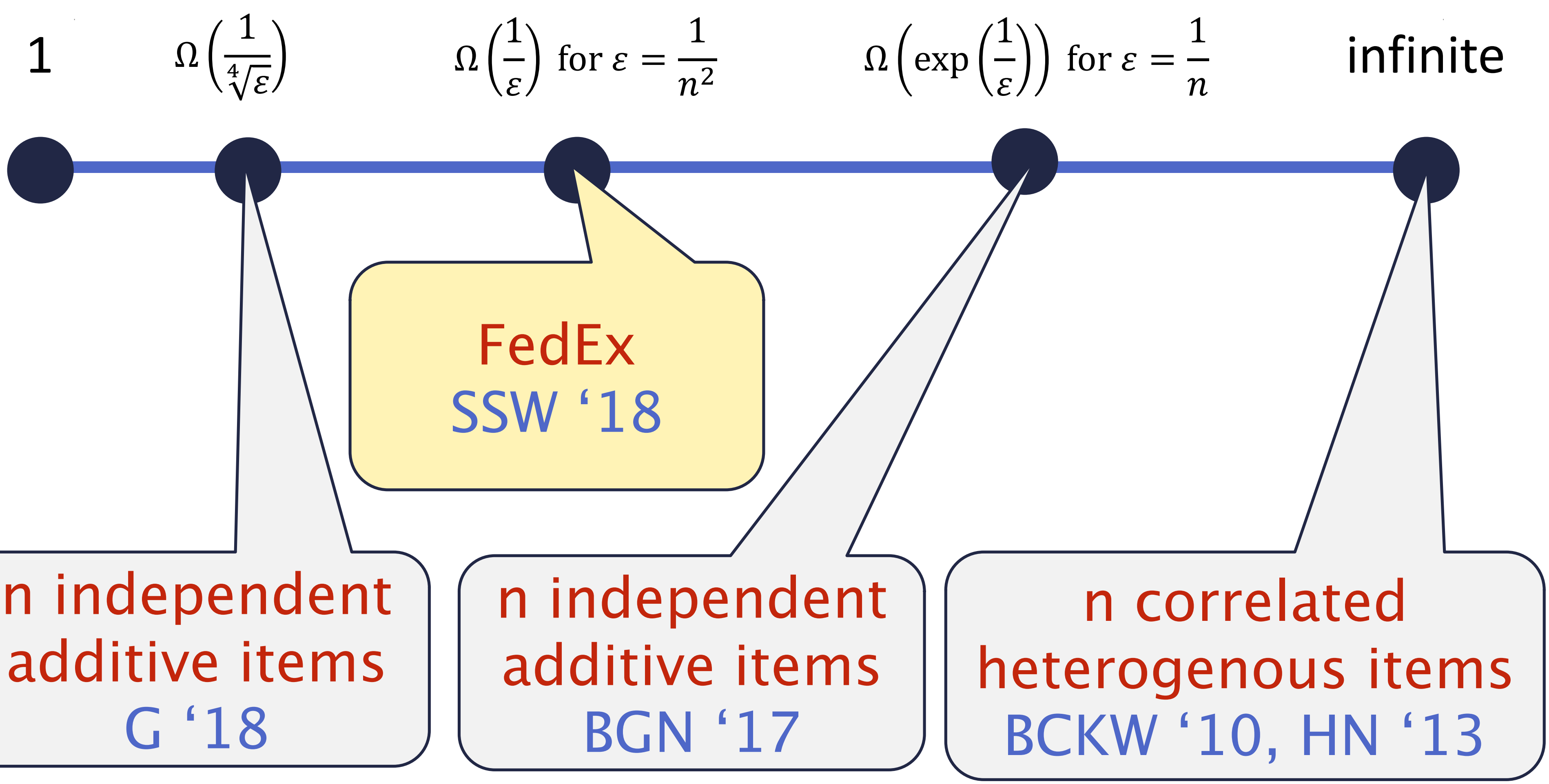
# Multi-Dimensional Menu Complexity for *n* Items



# Optimal Menu Complexity Spectrum



# Lower Bounds for $(1-\varepsilon)$-approximations



# To Come

The degree of complexity in the menu comes from the IC constraints which stitch together otherwise separate 1D problems.

Methods for understanding this:

- **Part I:** Revenue Curves
- **Part II:** Complementary Slackness conditions

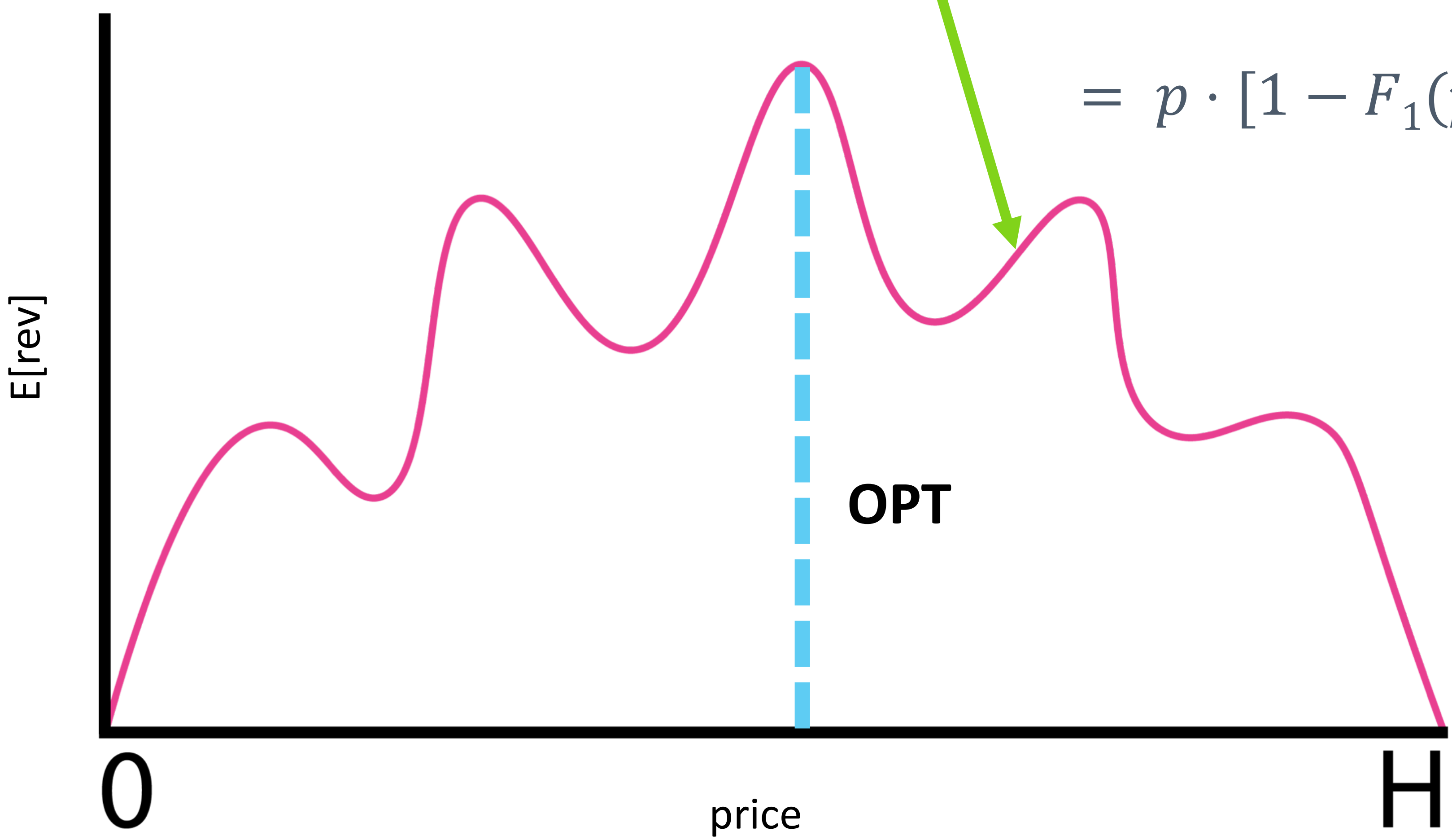
# Part I: Revenue Curves

METHODOLOGY FOR UNDERSTANDING THE NUMBER OF PRICES NEEDED

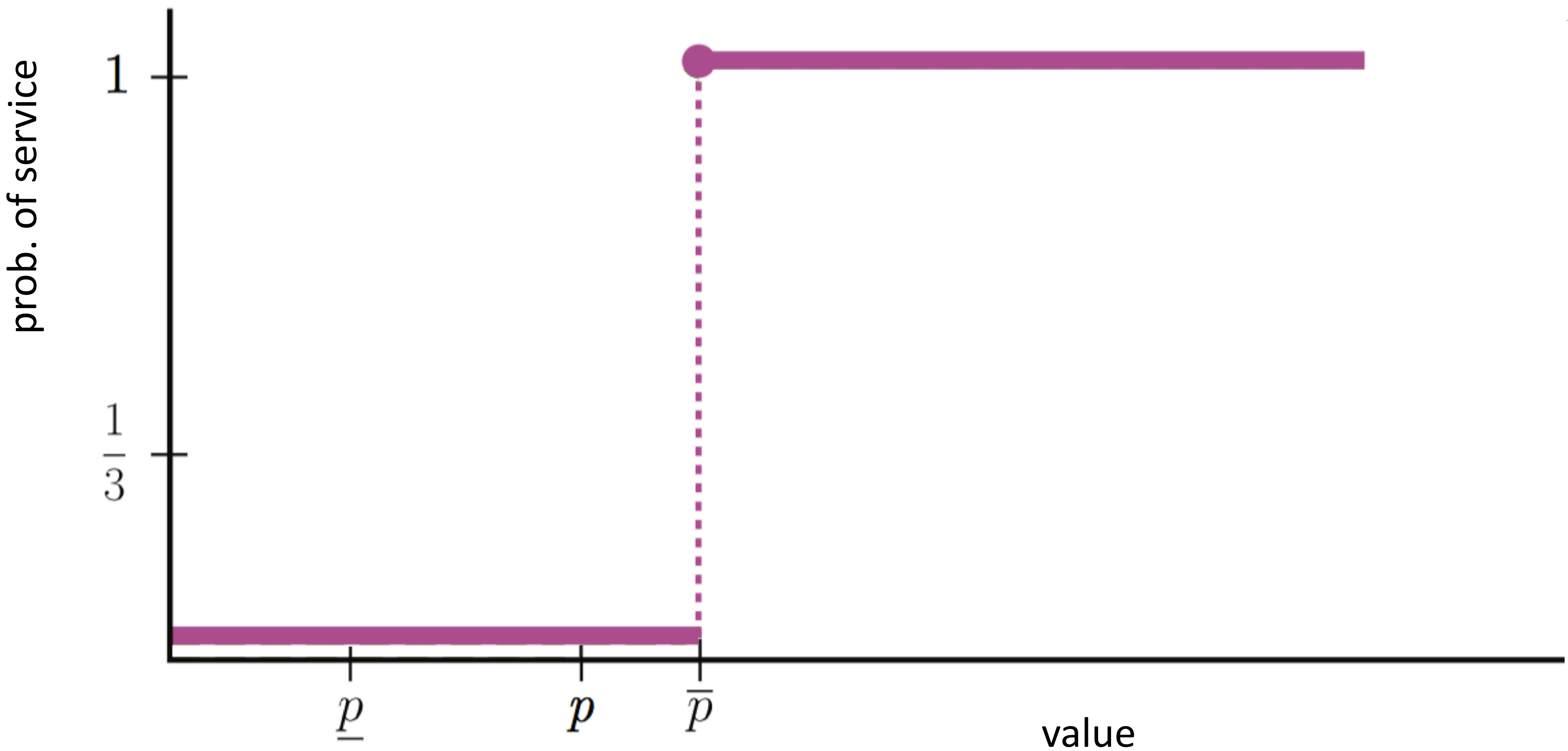
# Mapping Prices to Revenue

p * Pr[customer buys at price p]

$$= p \cdot [1 - F_1(p)]$$



# Allocation Rules and Prices



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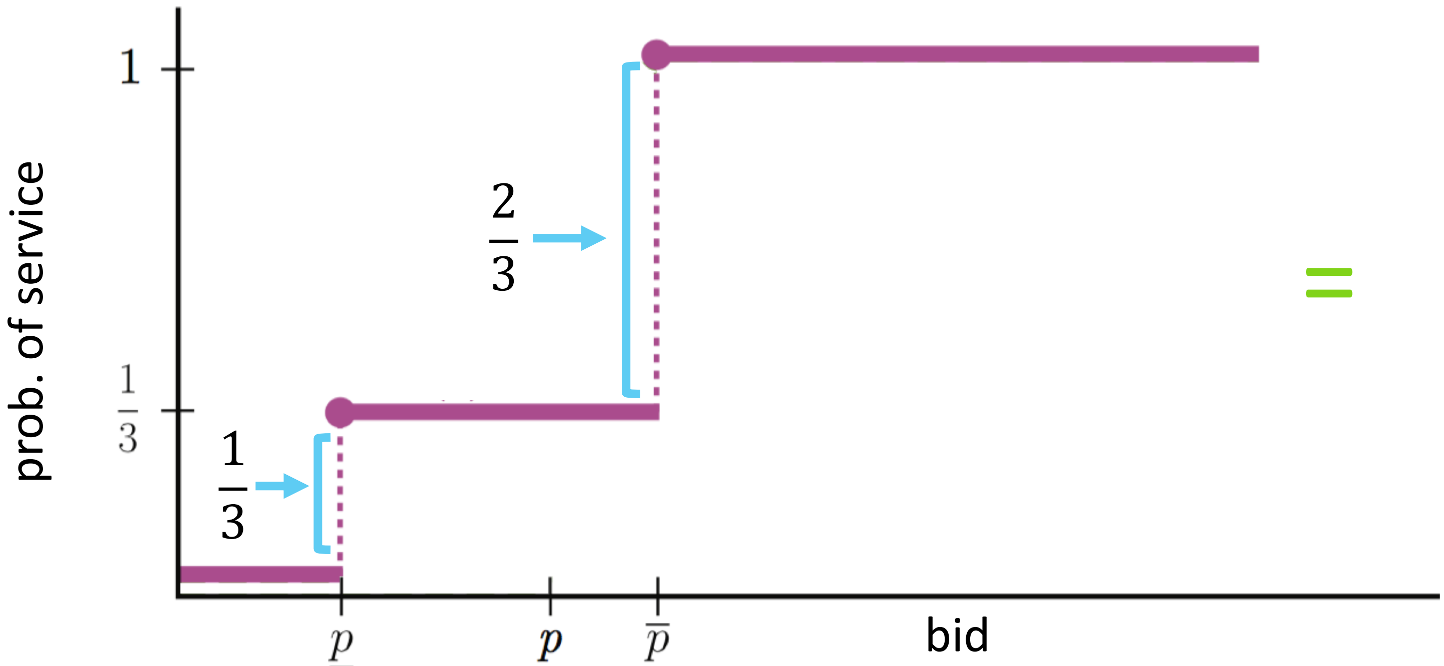



=

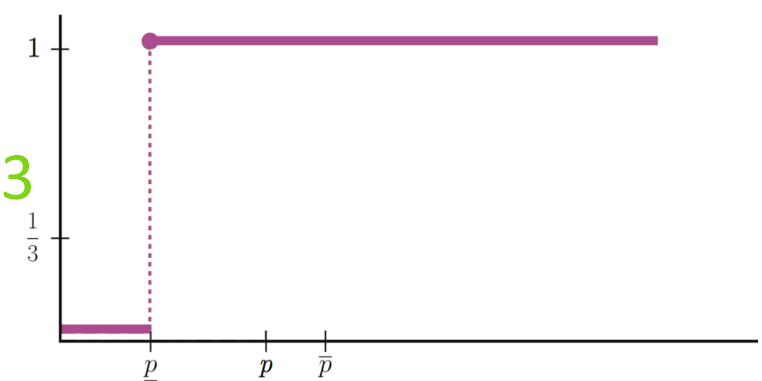
Menu

$$(1,\ \tfrac{1}{3}\underline{p} + \tfrac{2}{3}\overline{p})$$
$$(\tfrac{1}{3}\ ,\ \tfrac{1}{3}\underline{p})$$
$$(0\ ,\ 0)$$

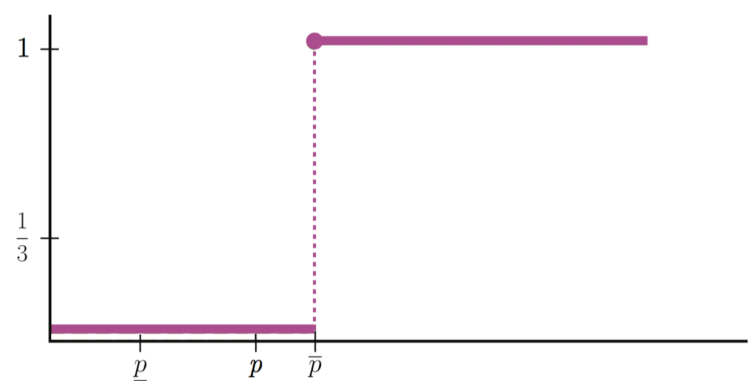
Menu Size = 2

= # prices in supp

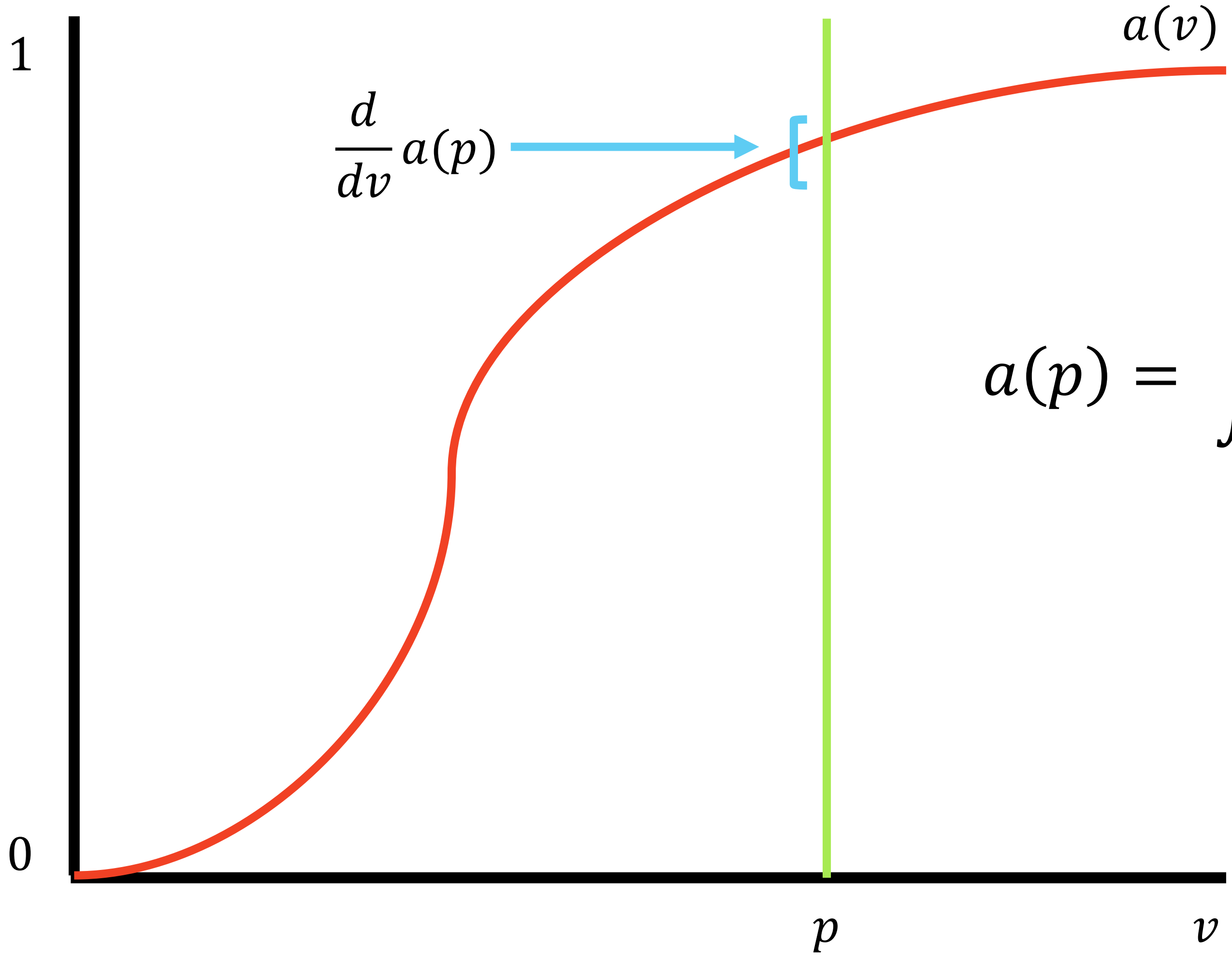
1/3



+ 2/3

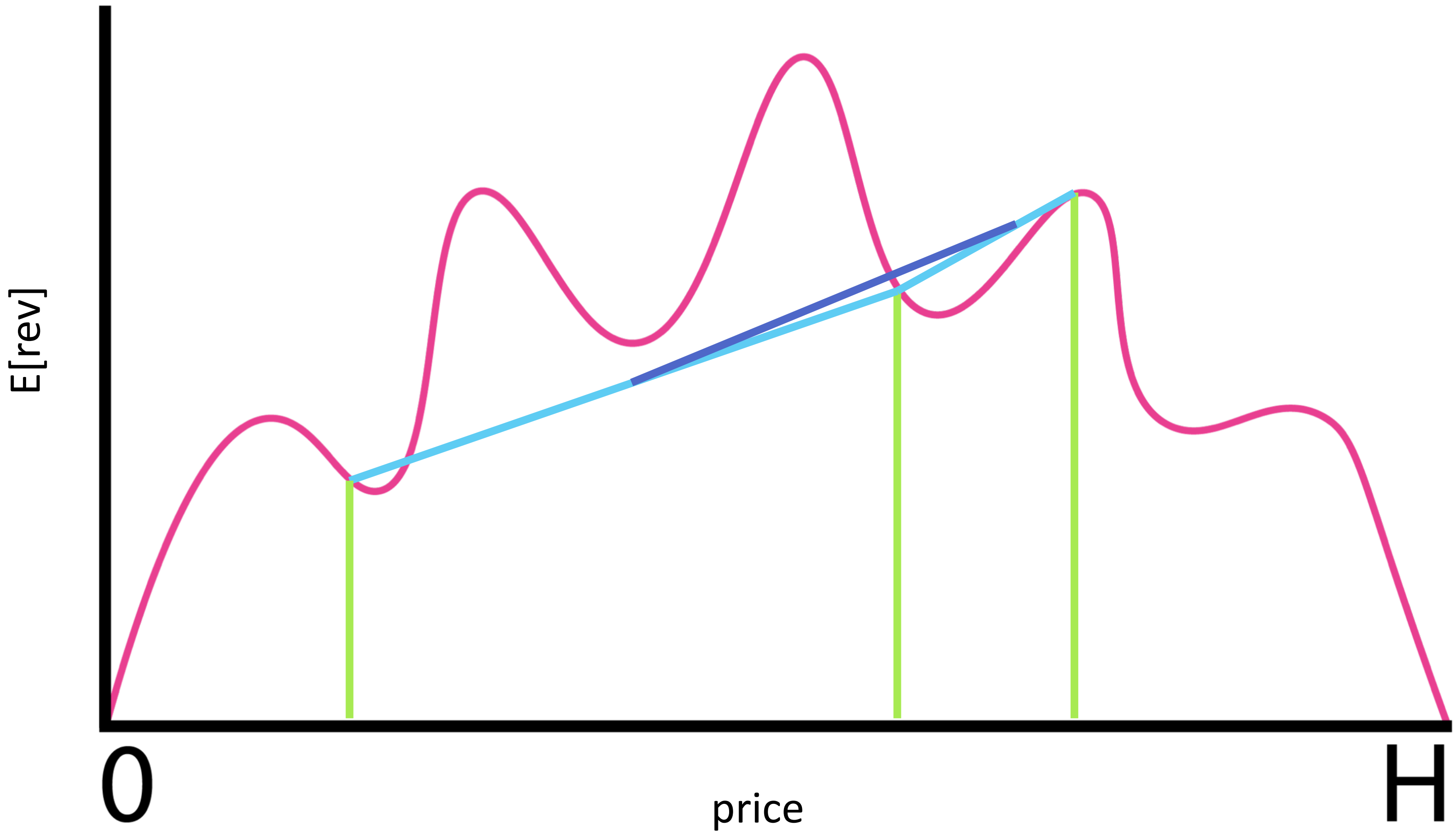


# Any allocation is a dist. over prices



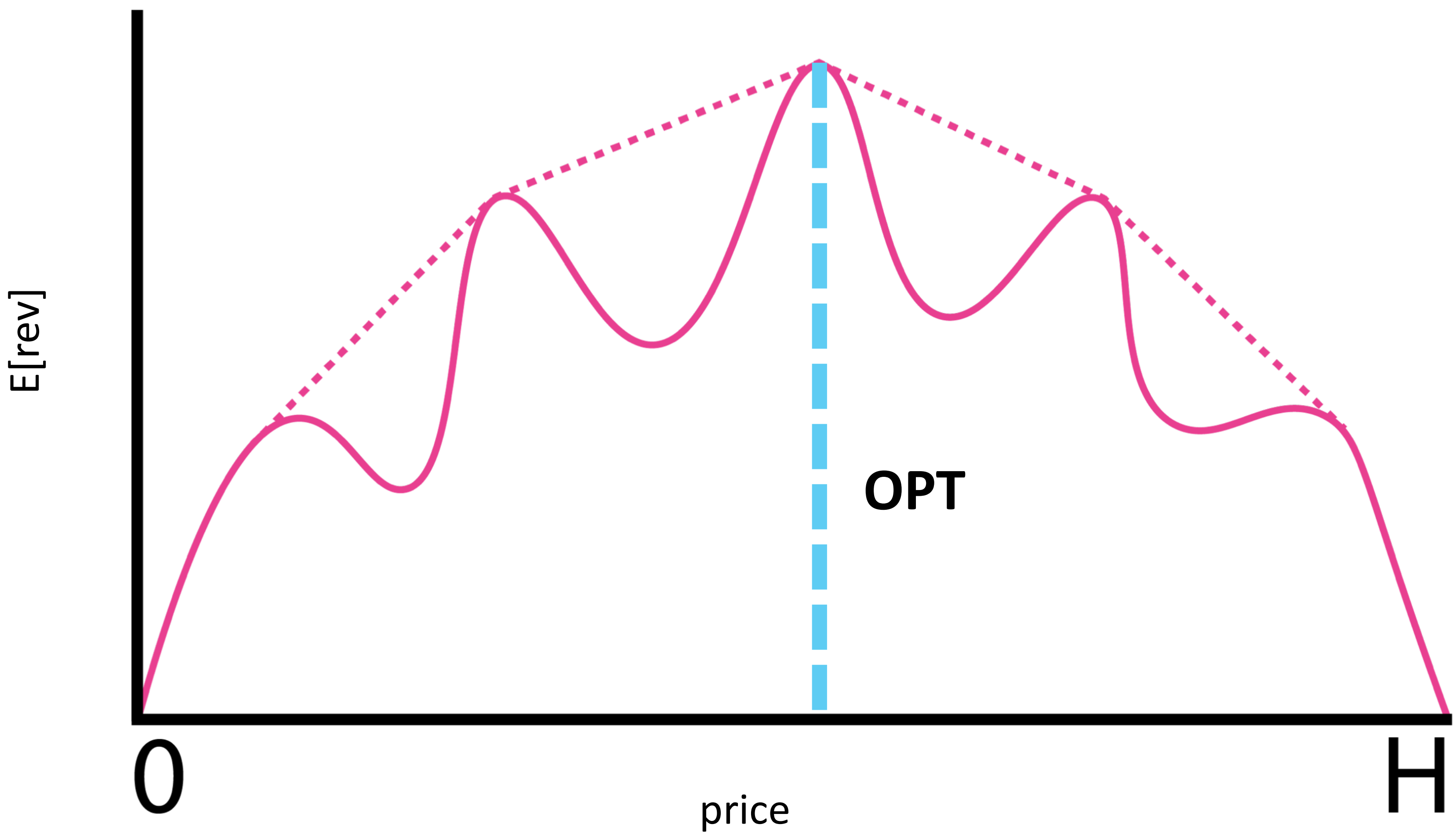

$$a(p) = \int_0^p \frac{d}{dv}a(v)dv$$

# Randomized Pricings



# “Ironed” Revenue Curve

Least concave upper bound on curve (in value space)



# The FedEx Setting

[FIAT GOLDNER KARLIN KOUSTOUPIAS 2016]

[Fiat G. Karlin Koutsoupias '16]

# The FedEx Setting


value $v$ =
how much shipping the package is worth

$(v,i) \sim F$


deadline $i$
= when they need the package sent by

*indifferent if the package is shipped early

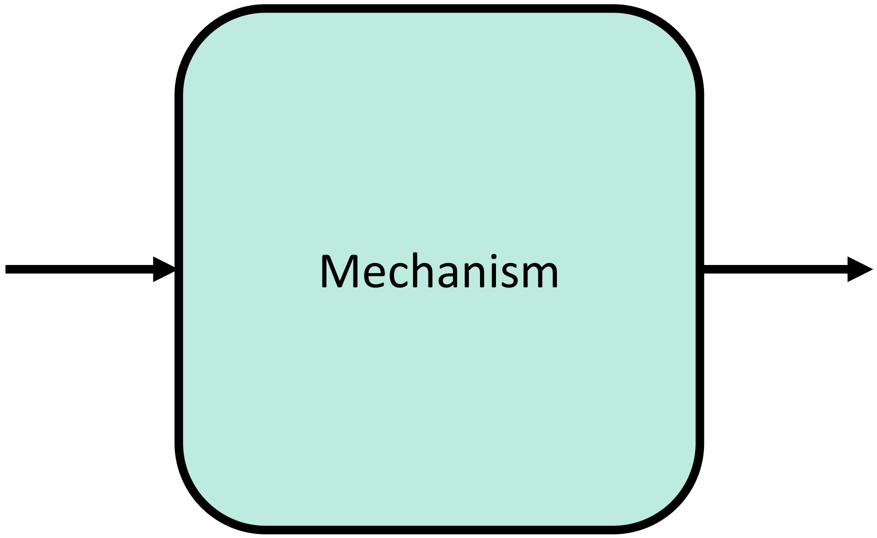
**Shipping options**

1 day ← 2 days ← 3 days ← ... ← $n$ days

[Fiat G. Karlin Koutsoupias '16]

# The FedEx Setting

$(v, d)$




$a_i(v, d)$ = prob buyer gets i-day shipping

$p(v, d)$ = expected payment

Can have *exponentially* many menu options

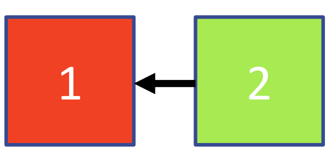
$(v, d) \sim F$

Menu

$(1 \cdot 1, \$5)$

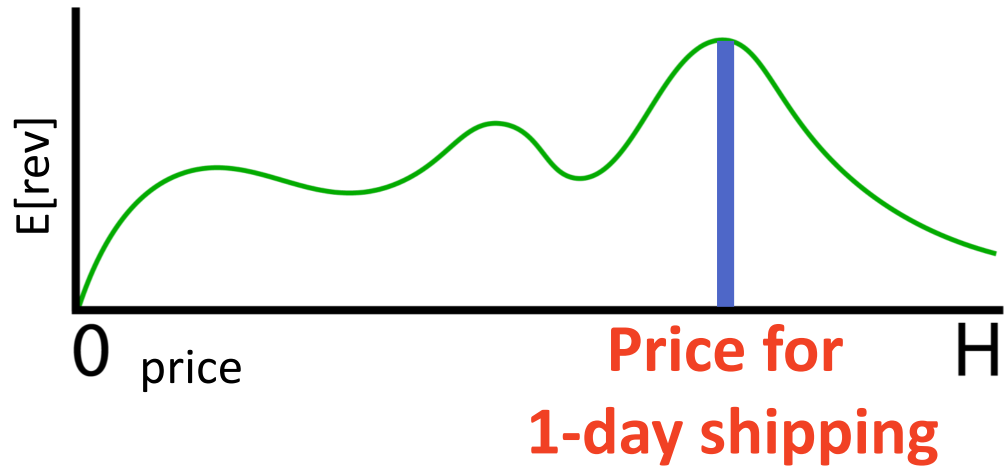
$(2 \cdot \frac{2}{3}, \$3)$

$(2 \cdot 1, \$5)$



# How do we maximize revenue for 2 days?

Day 1
revenue curve

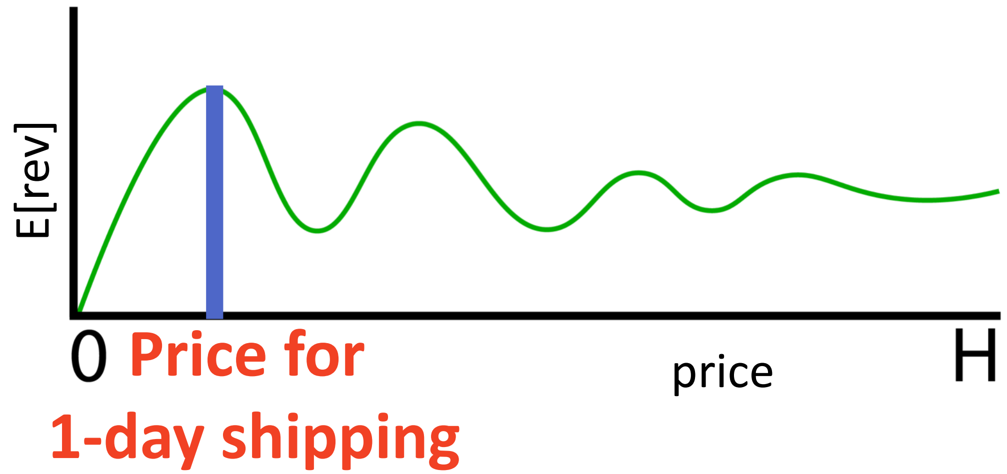



Day 2
revenue curve


Set best prices for each day when they are decreasing.

# How do we maximize revenue for 2 days?

Day 1
revenue curve

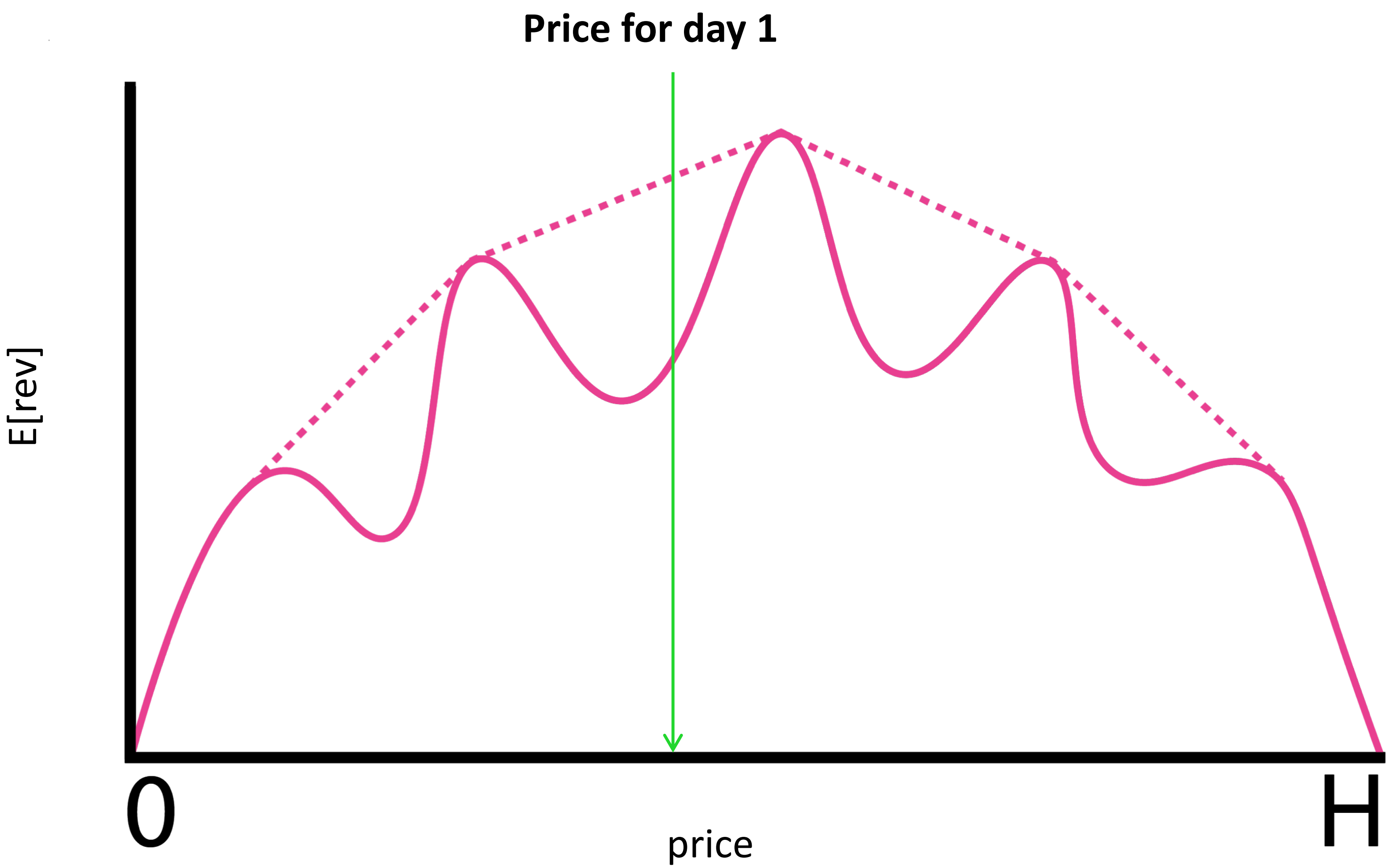



Day 2
revenue curve


Not incentive compatible!

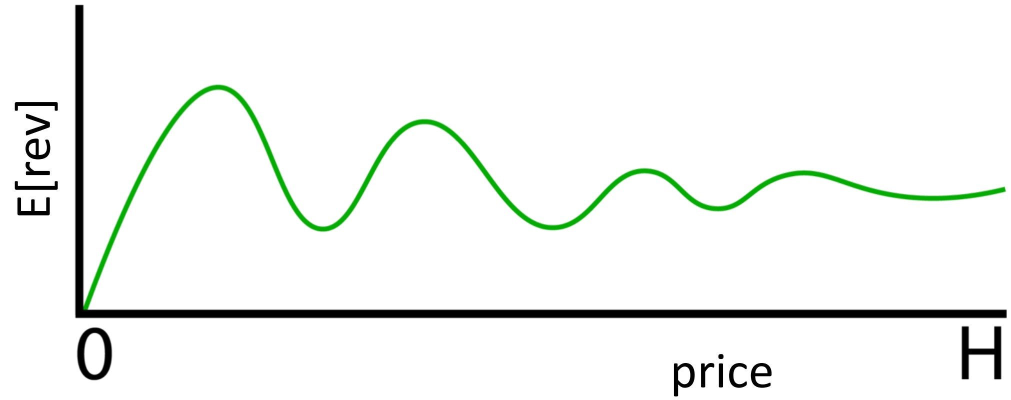
# FedEx
# Revenue Curves

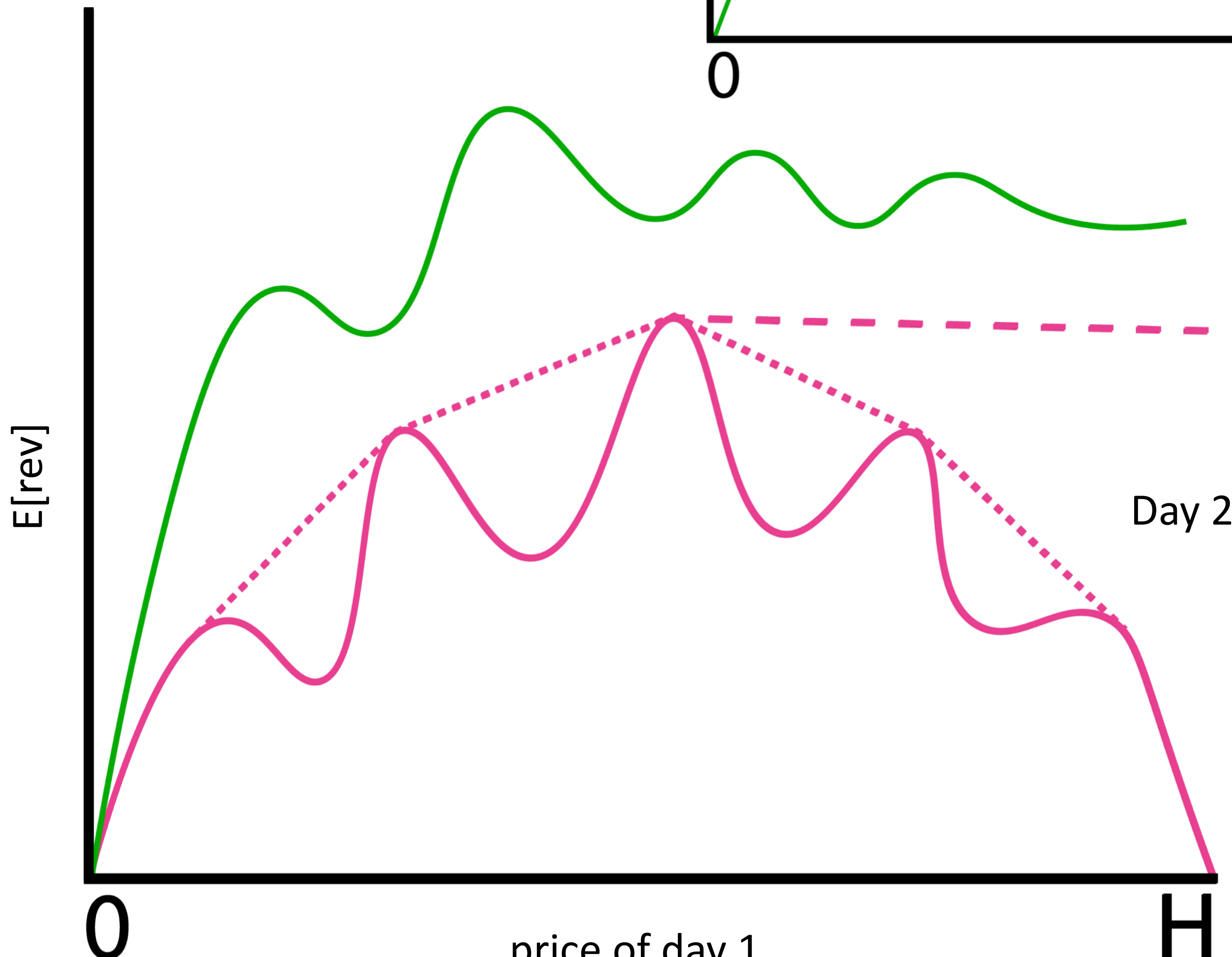
# Constrained revenue from Day 2:



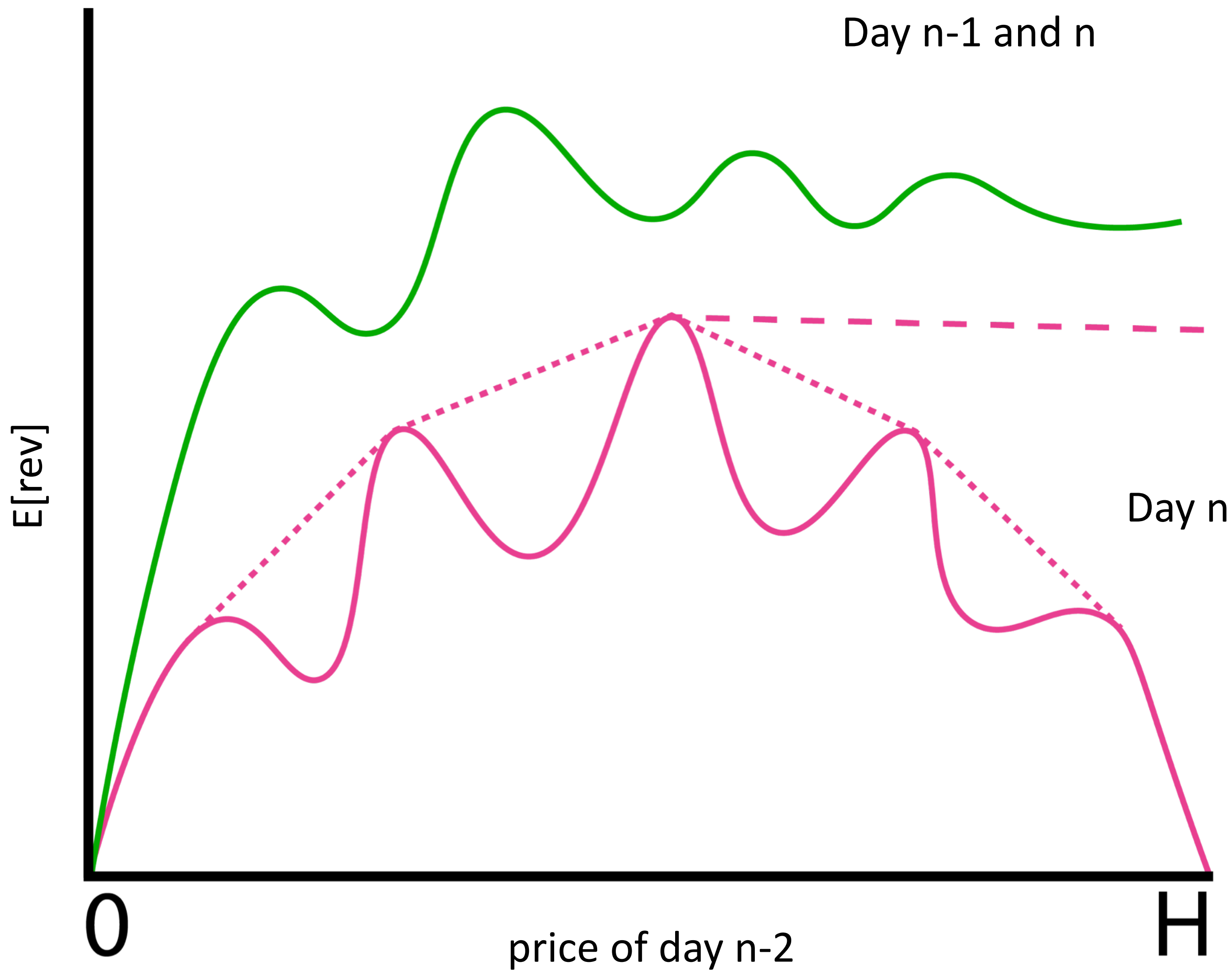
# Constrained revenue from Day 2:

# What to optimize:

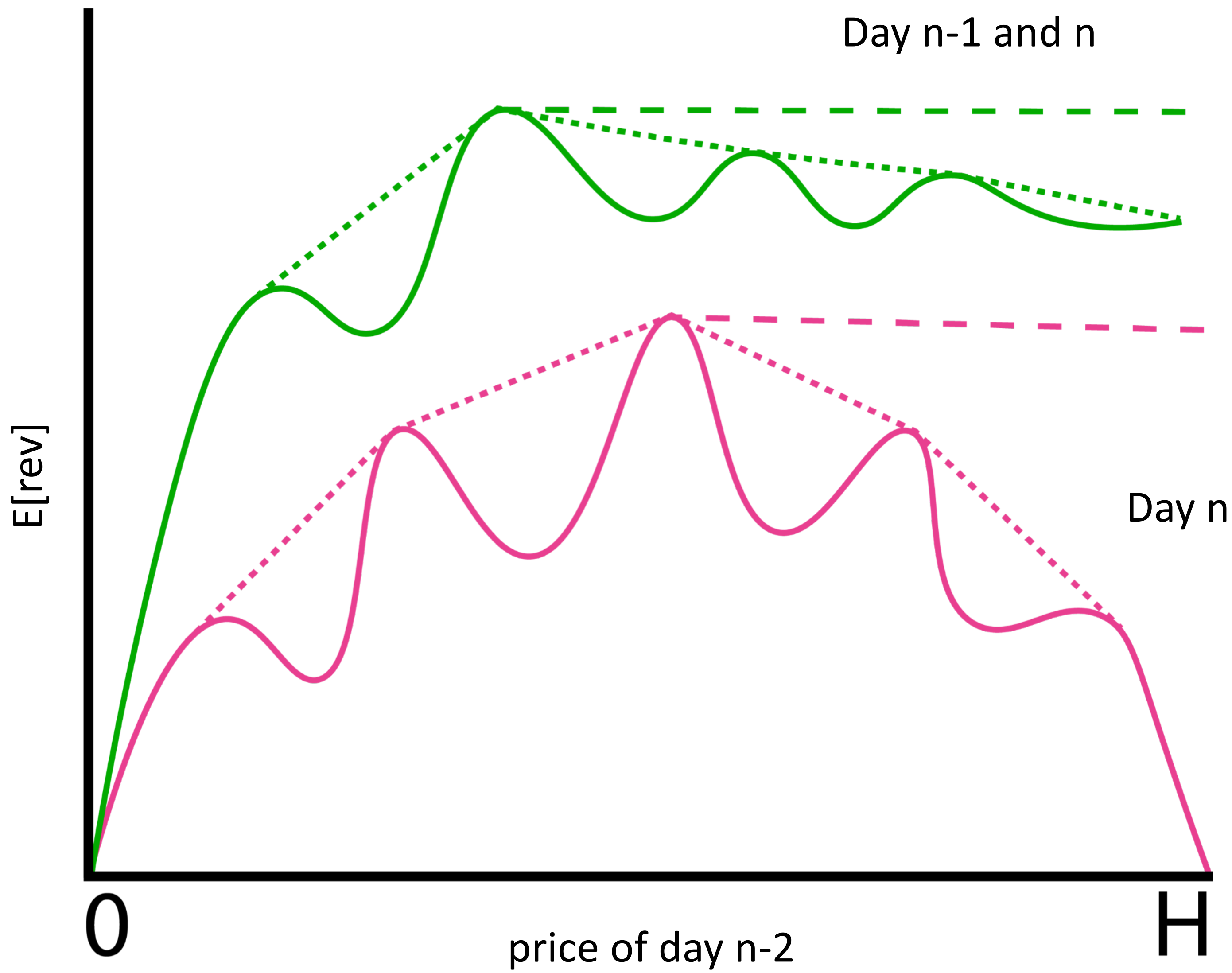


# What to optimize:

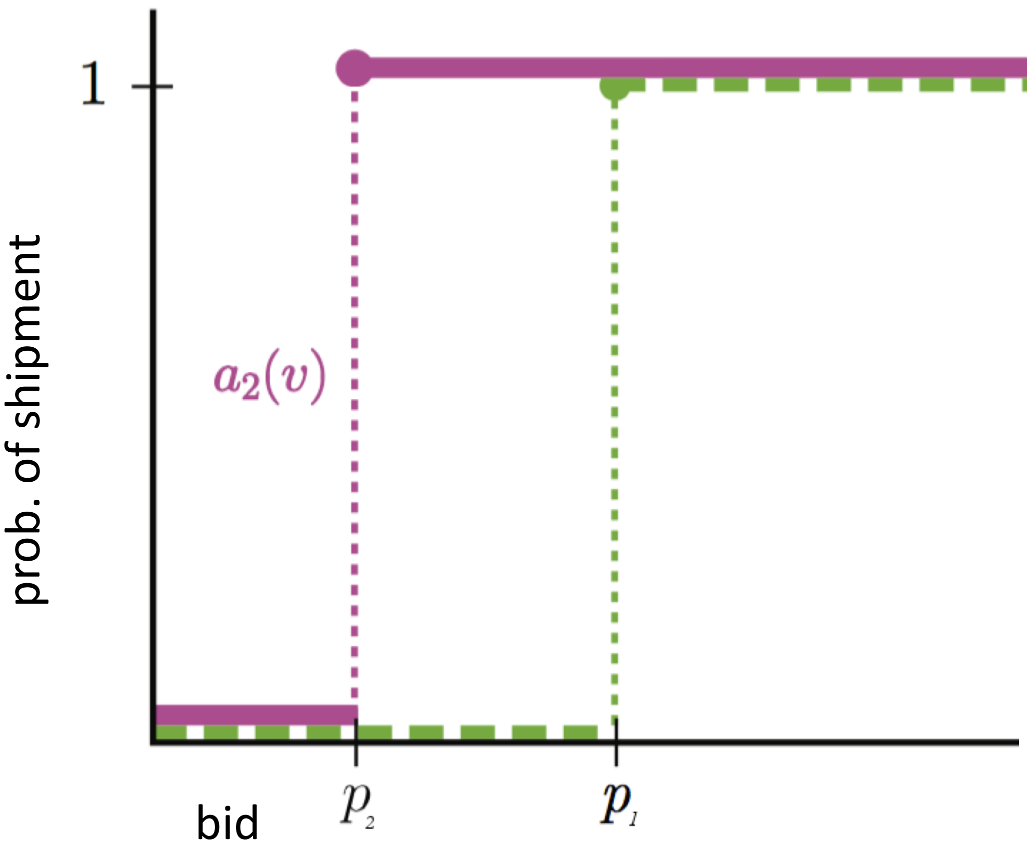


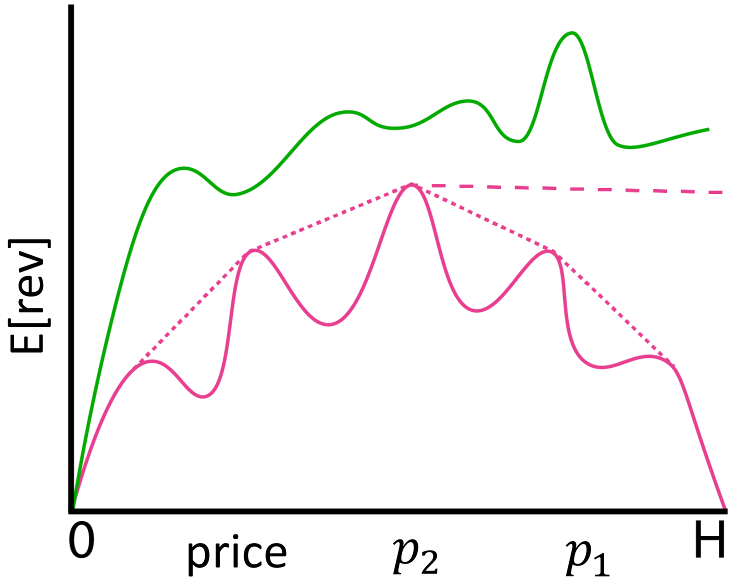
# What to optimize:



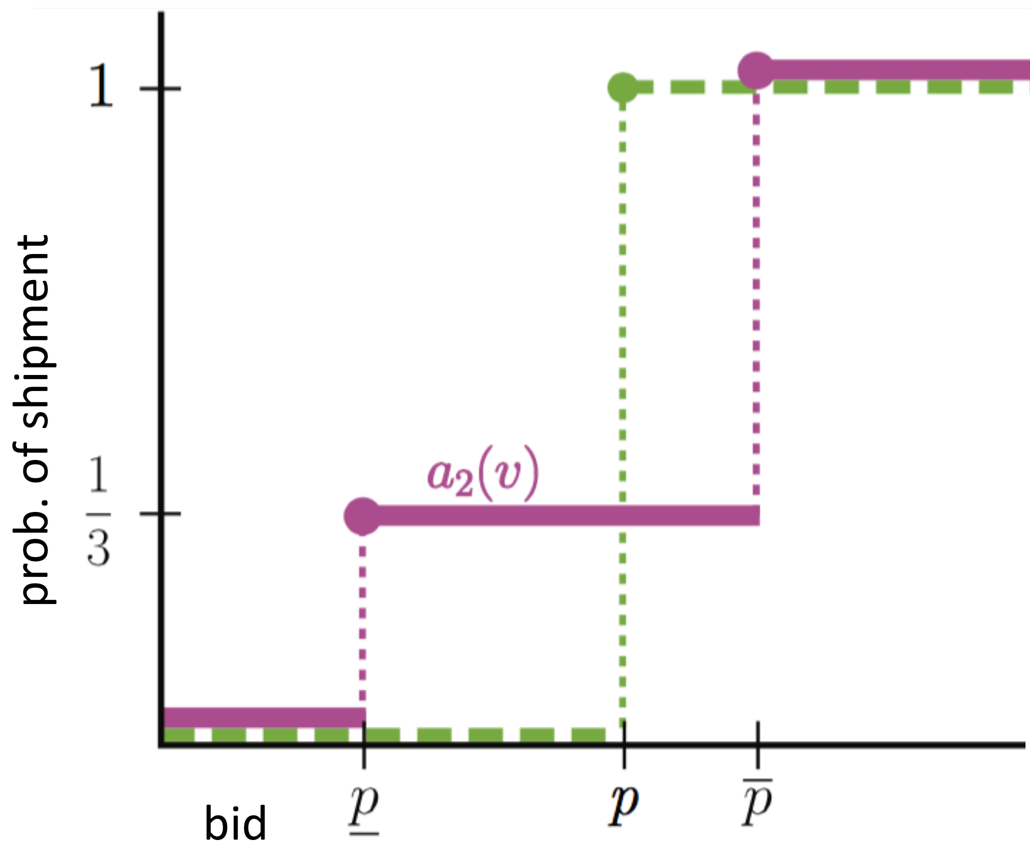
# Optimal Variables

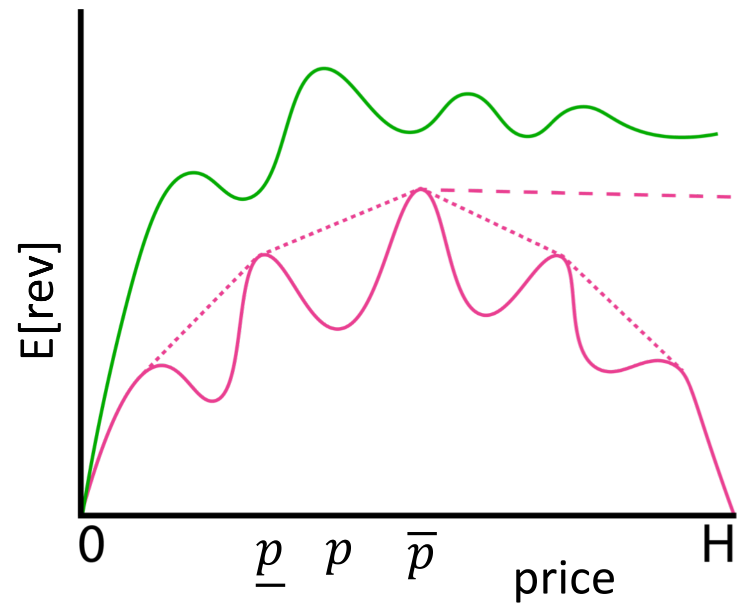
# Optimal Allocation Rule



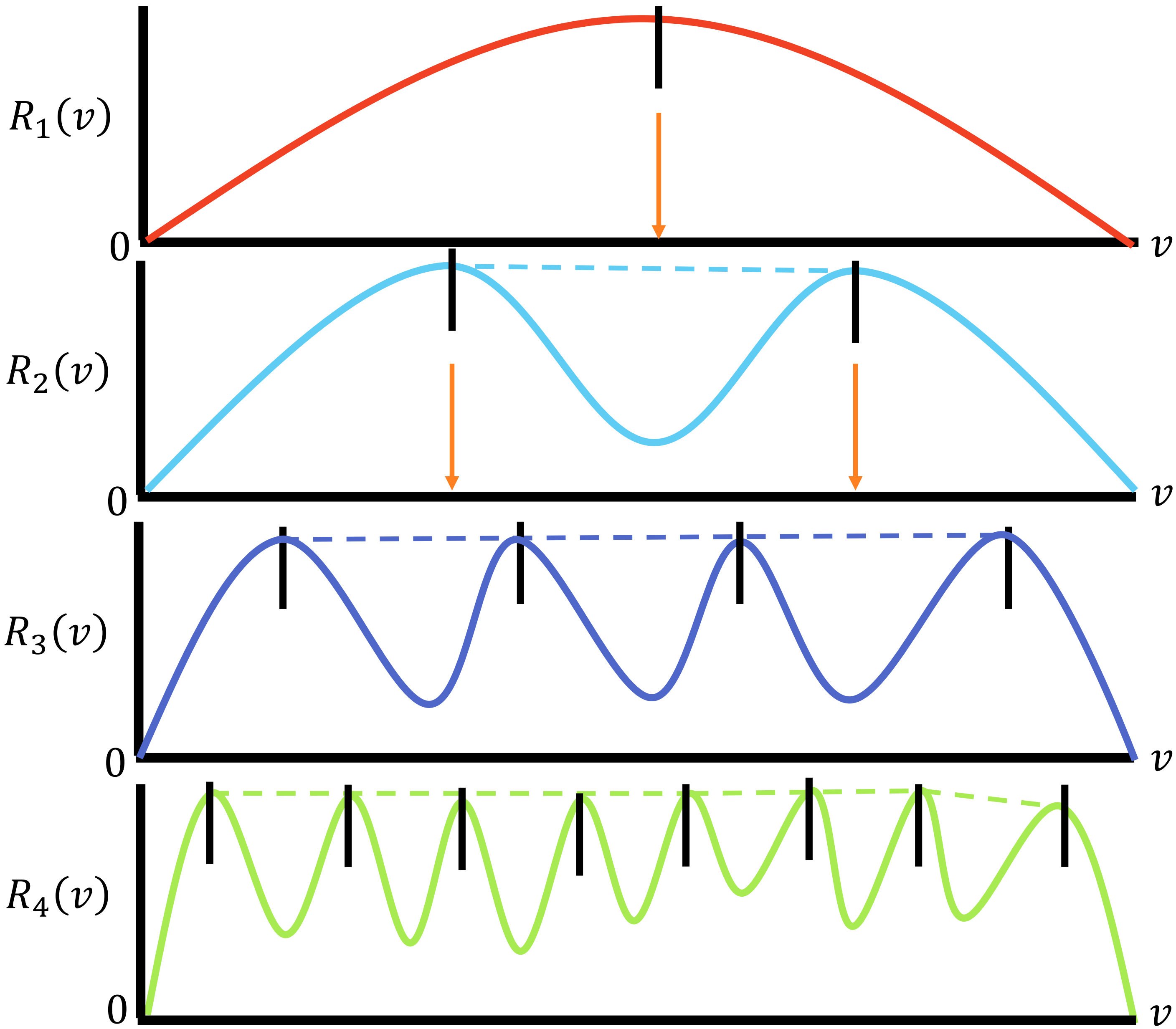



# Optimal Allocation Rule



# Bad Example

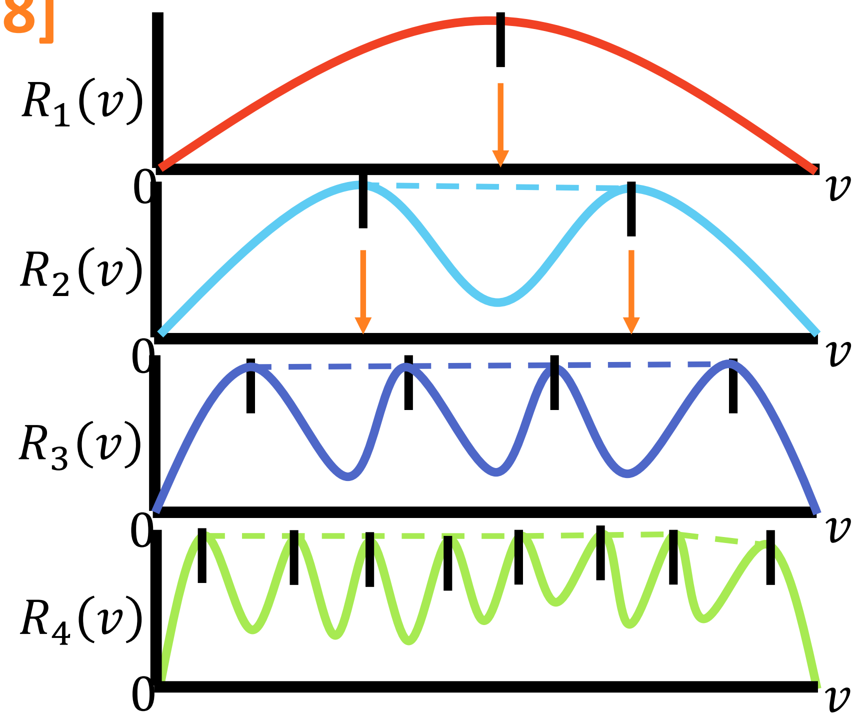


# Exponential Menu Complexity

**Upper Bound:** In the worst case, each deadline $i$ has $2^{i-1}$ options. **[Fiat G. Karlin Koutsoupias '16]**

**Lower Bound:** Distributions exist for this example, forcing $2^{i-1}$ options for each deadline.
**[Saxena Schvartzman Weinberg '18]**

Menu size is $2^n - 1$ overall, tight.

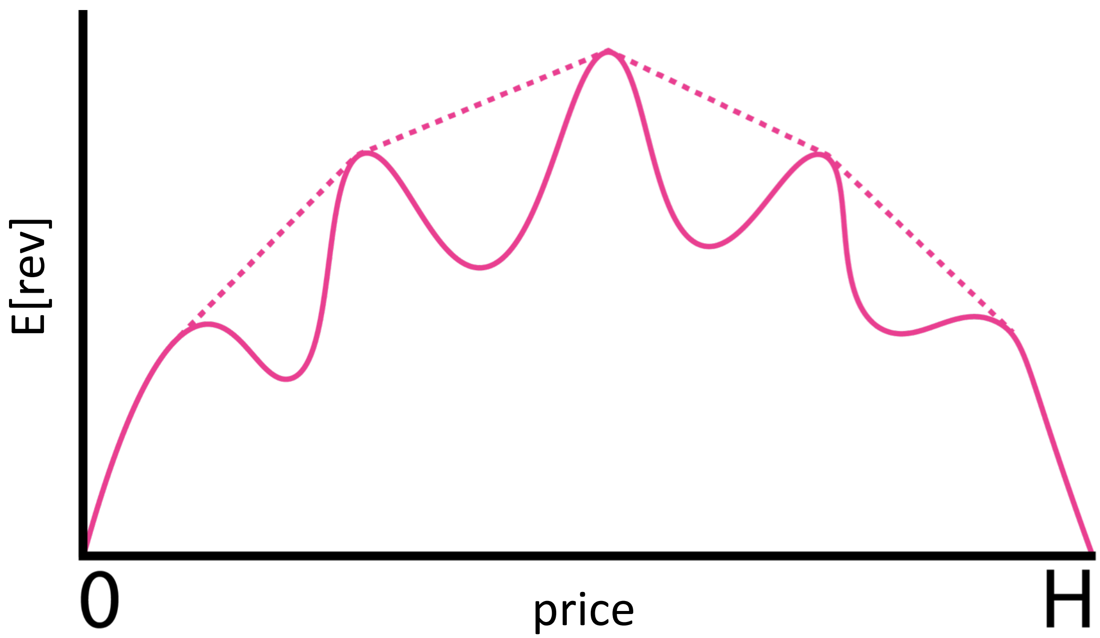


# Approximate FedEx Menu Complexity

[SAXENA SCHVARTZMAN WEINBERG 2018]

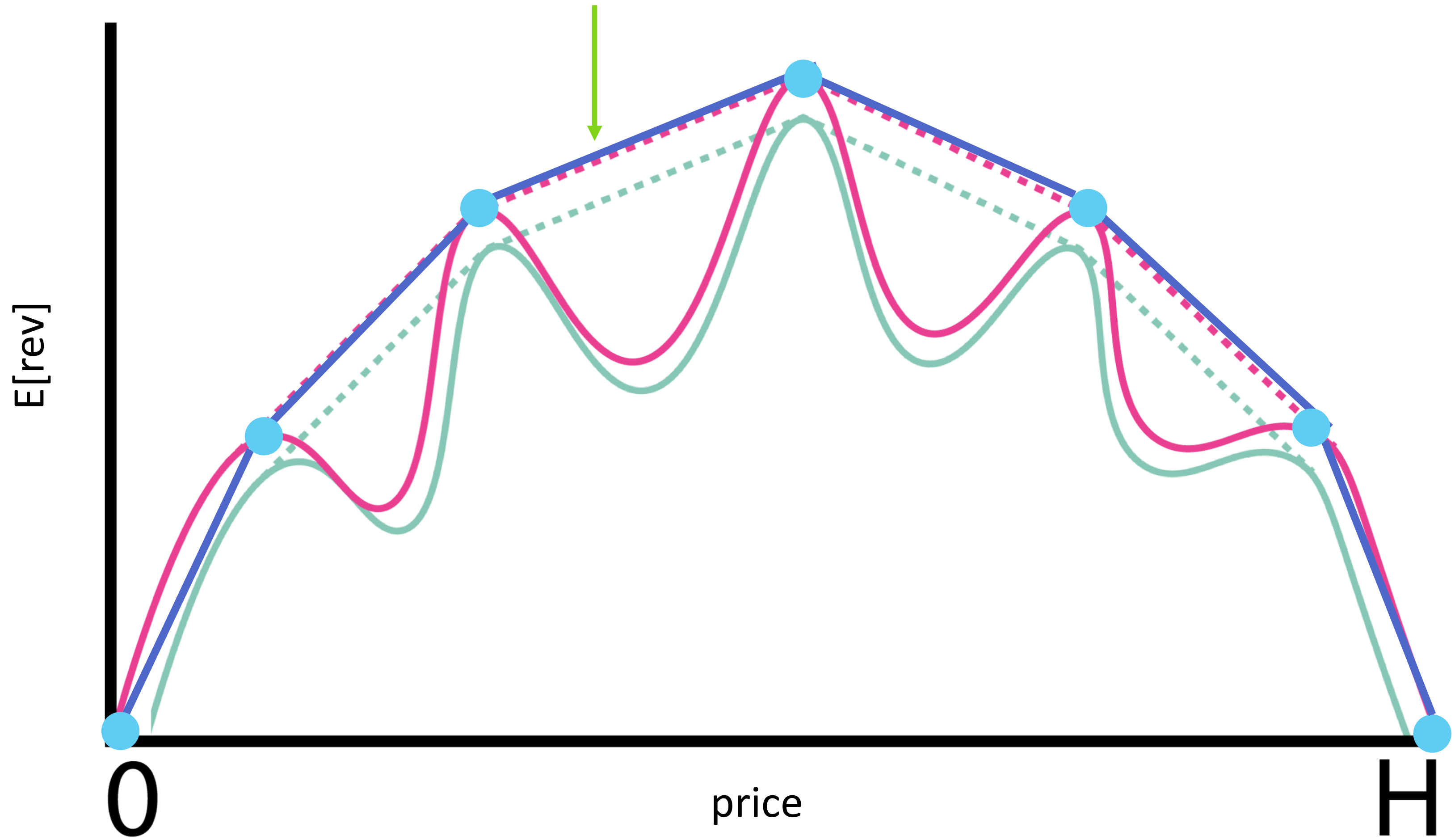
# Limiting Menu Complexity

How can we achieve good revenue with a **small menu**, or equivalently randomizing over **fewer prices**?




**Idea:** We only randomize over **un-ironed peaks**.
What if we constrain this number?

# Revenue via Polygon Approximation



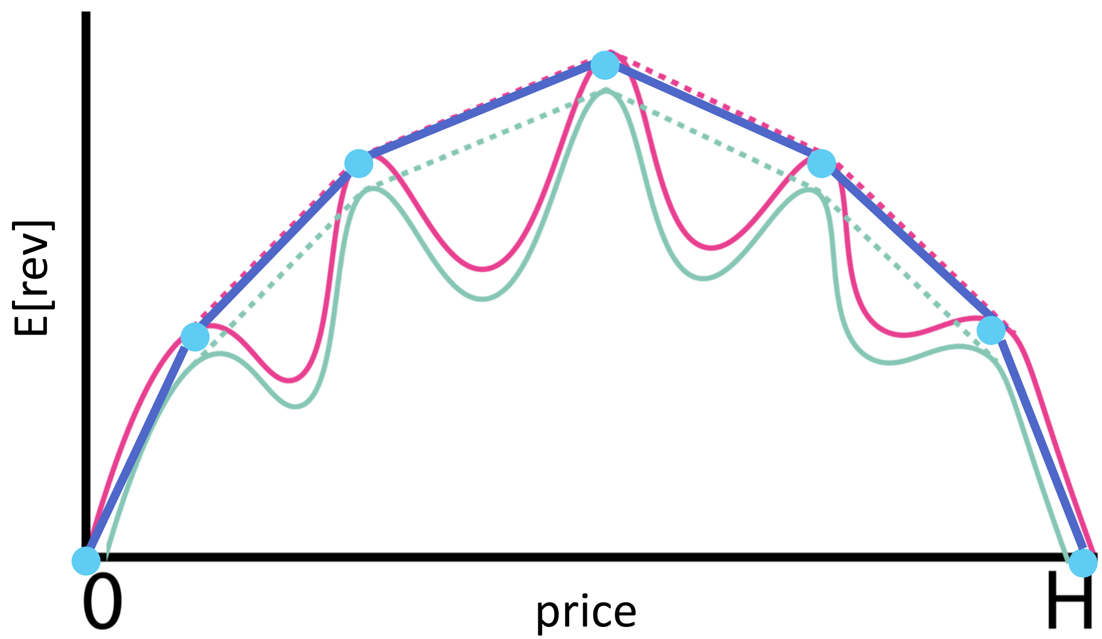

Menu size is limited by the # points supporting the curve.

[Saxena Schvartzman Weinberg '18]

# Menu Complexity for $(1-\varepsilon)$-approx

Upper Bound: $O\left(n^{\frac{3}{2}}\sqrt{\frac{\min\left\{\frac{n}{\varepsilon}, \ln(H)\right\}}{\varepsilon}}\right) = O\left(\frac{n^2}{\varepsilon}\right)$

Lower Bound: $\Omega(n^2) = \Omega\left(\frac{1}{\varepsilon}\right)$ for $\varepsilon = O\left(\frac{1}{n^2}\right)$



# Revenue Curve Recap

- **Splitting** into multiple prices originates from IC constraints.
- Curves depict the **limits** of how prices can split.
- Essentially any combination of peaks/valleys **can exist**. **[Saxena Schvartzman Weinberg '18]**
- When the mechanism is determined by revenue curves, approximation can be done via **revenue curve approximation**.

# Part II: Duality Approach

METHODOLOGY FOR REASONING ABOUT WHEN ALLOCATION PROBABILITIES MUST BE DISTINCT

# The Primal

Maximize E[Rev]

subject to:

more utility for (v,i) than (v',i')

feasibility

---

Maximize **E[Virtual Welfare]**

subject to:

more utility for (v,i) than (v,i')

weak monotonicity of allocation

feasibility

# Duality

**Primal**

maximize $f(\mathbf{x})$

subject to $g(\mathbf{x}) \geq 0$

**Dual**

minimize $h(\mathbf{y})$

subject to $r(\mathbf{y}) \leq 0$

Optimal pair $(\mathbf{x},\mathbf{y})$ $\Leftrightarrow$ complementary slackness is satisfied, feasible: $g(\mathbf{x}) = 0$ or $\mathbf{y} = 0$; $h(\mathbf{y}) = 0$ or $\mathbf{x} = 0$.

**Lagrangian Primal:** $\text{maximize}_{\mathbf{x}}\ \text{minimize}_{\mathbf{y}}\ f(\mathbf{x}) + \mathbf{y}\, g(\mathbf{x})$.

**Lagrangian Dual:** $\text{minimize}_{\mathbf{y}}\ \text{maximize}_{\mathbf{x}}\ f(\mathbf{x}) + \mathbf{y}\, g(\mathbf{x})$.

Complementary slackness: $g(\mathbf{x}) = 0$ or $\mathbf{y} = 0$.

# The Primal

$a_i(v) :=$ Pr[$i-$day shipping to bidder with $(v, i)$]

maximize

$$\sum_i \int_0^H f_i(v)\varphi_i(v)a_i(v)\,dv$$

= E[rev$_i$] using payment identity

subject to:

Dual variables

| | | | |
|---|---|---|---|
| $\int_0^v a_i(x)dx - \int_0^v a_{i-1}(x)dx \geq 0$ | Report i over i' | $\alpha_{i,i-1}(v)$ | $\forall i \in \{2, \ldots, n\}$ |
| $a_i'(v) \geq 0$ | Report v over v' | $\lambda_i(v)$ | $\forall i, v$ |
| $a_i(v) \in [0, 1]$ | feasibility | | |

# The Dual

$a_i(v) \coloneqq \Pr[i\text{–day shipping to bidder with } (v, i)]$

minimize $_{\lambda,\alpha}$ maximize $_{\text{feasible } a}$

$$\sum_i \int_0^v f_i(v) a_i(v) \Phi_i(v) dv$$

where

$$\Phi_i(v) \coloneqq \varphi_i(v) + \frac{\left(\int_v^H \alpha_{i,i-1}(x)\, dx - \int_v^H \alpha_{i+1,i}(x) dx\right) - \lambda_i'(v)}{f_i(v)}$$

# An Optimal Primal/Dual Pair

minimize $_{\lambda,\alpha}$ maximize $_{\text{feasible } a}$

$$\sum_i \int_0^H f_i(v)\boldsymbol{a_i(v)}\,\boldsymbol{\Phi_i(v)}\,dv$$

Can't change $\lambda$,$\alpha$ to further minimize.

## Complementary Slackness:

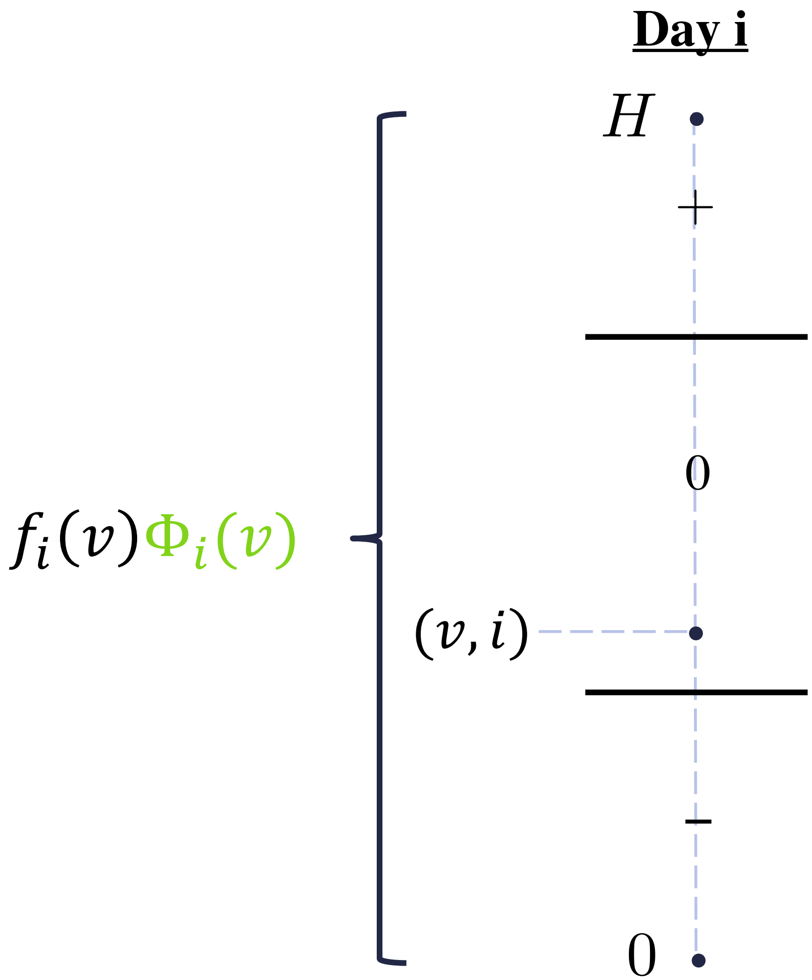
**Constraint** is tight (= 0) or **dual variable** is 0.

Dual variables

| | | |
|---|---|---|
| Report i over i' | $\int_0^v a_i(x)dx - \int_0^v a_{i-1}(x)dx \geq 0$ | $\boldsymbol{\alpha_{i,i-1}(v)}$ |
| Report v over v' | $a_i'(v) \geq 0$ | $\boldsymbol{\lambda_i(v)}$ |

# Understanding Dual Variables

# Virtual Values




$$\min_{\lambda,\alpha}\max_{a}\sum_i \int_0^H f_i(v)\Phi_i(v)a_i(v)\,dv$$

$a_i(v) = 1$

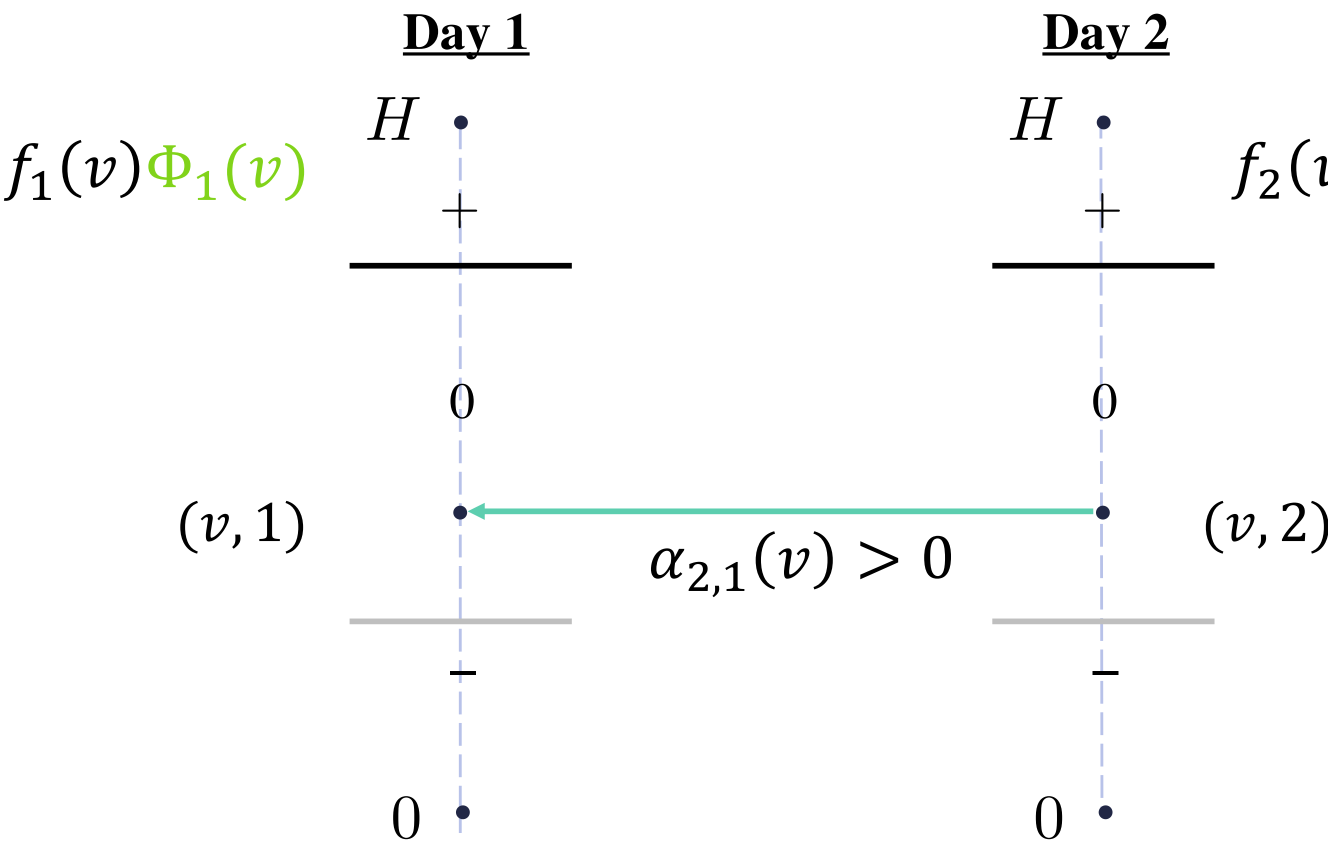
$a_i(v) = ??$

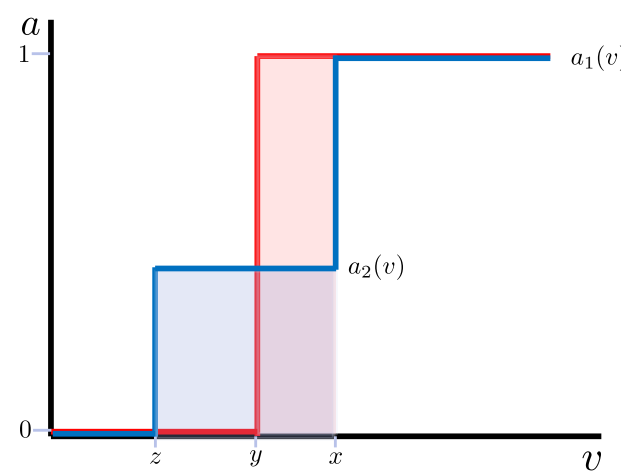
$a_i(v) = 0$

It's left to us to determine the allocation in the zeroes to satisfy complementary slackness.

# Dual Variable $\alpha$ (reporting i over i-1)

$$\int_0^v a_i(x)dx - \int_0^v a_{i-1}(x)dx \geq 0$$ Report i over i-1 $\boldsymbol{\alpha_{i,i-1}(v)}$

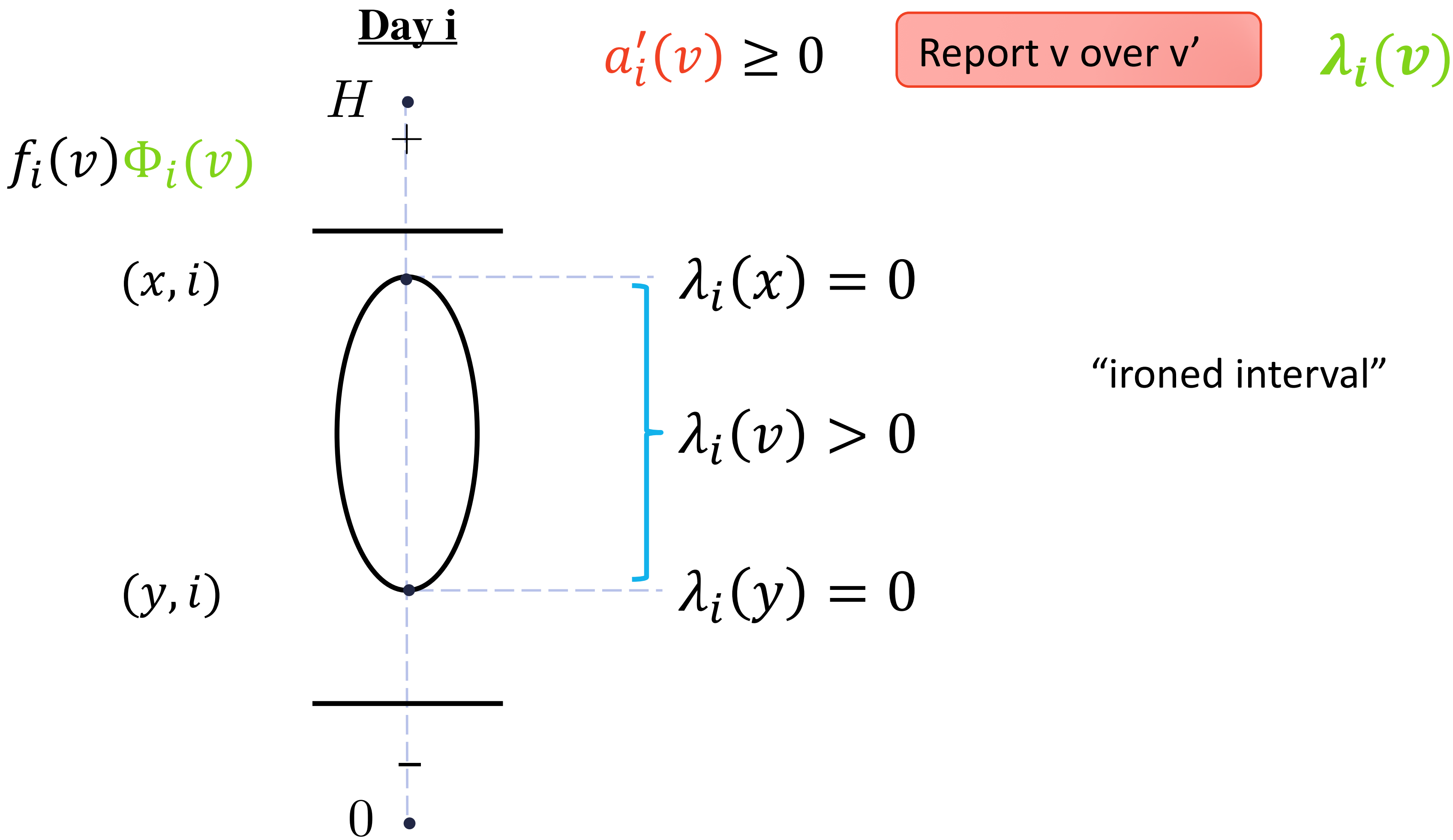



**Complementary Slackness:**

Inter-day utility is equal ($u_1 = u_2$) where $\alpha_{2,1}$ is positive.

# Dual Variable $\lambda$: (reporting v over v')




**Complementary Slackness:**

Utility is equal for reporting just under v—$a_i'(v) = 0$.

**The allocation is constant in ironed intervals: $a_i(v) = a_i(y)$.**

# Recap

Because $\boldsymbol{a}$ maximizes VW,
$\boldsymbol{\Phi}_i(v) > 0 \implies \boldsymbol{a}_i(v) = 1$ and $\boldsymbol{\Phi}_i(v) < 0 \implies \boldsymbol{a}_i(v) = 0$

Complementary slackness with $\lambda$:
$\lambda_i(v) > 0$ means $v$ is in an ironed interval $[\underline{v}, \overline{v}]$ and implies $\boldsymbol{a}_i(v)$ is constant on $[\underline{v}, \overline{v}]$ , or $\boldsymbol{a}_i(v) = \boldsymbol{a}_i(\underline{v})$.

Complementary slackness with $\alpha$:
$\alpha_{i,i-1}(v) > 0$ implies **utility** is equal across deadlines i, i-1

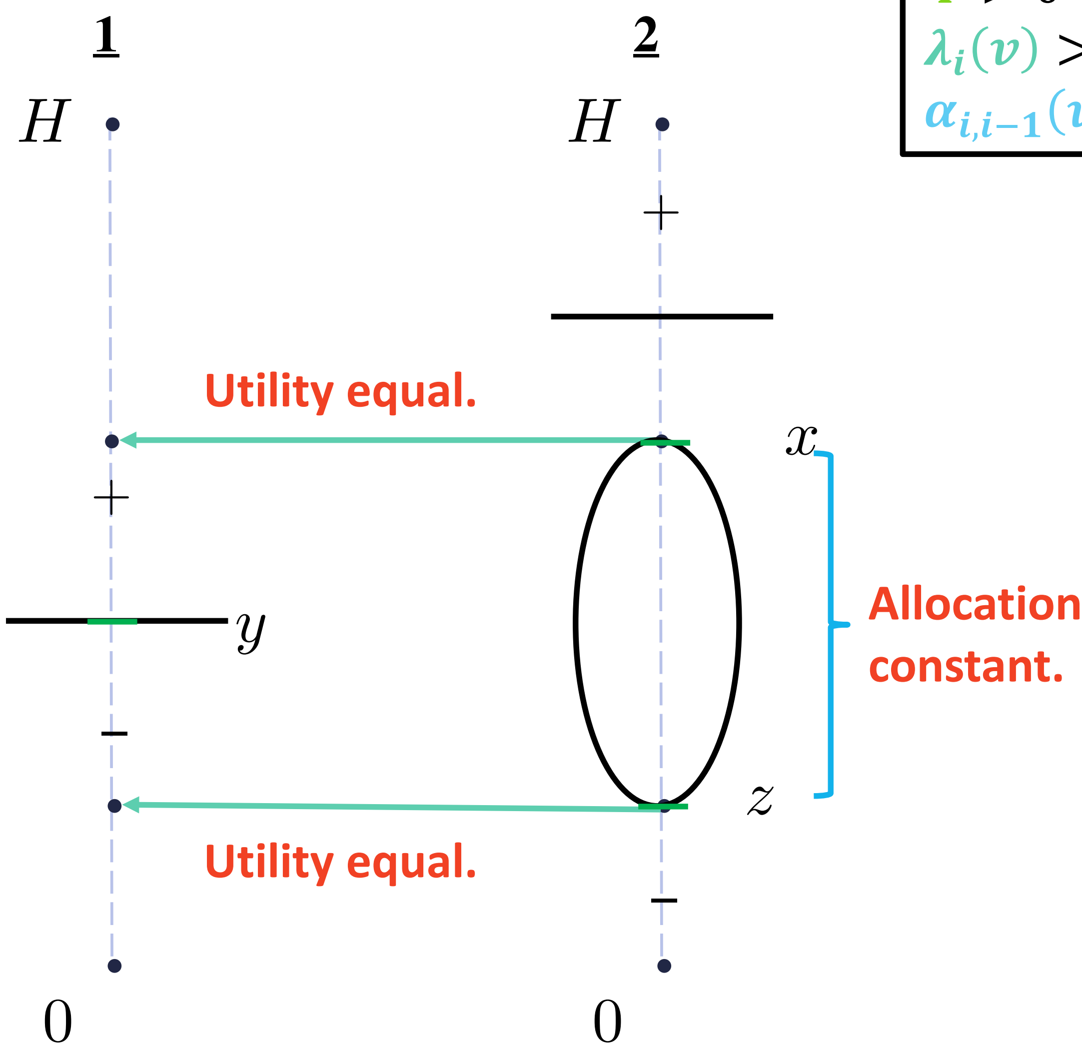
$\Phi > 0 \implies a = 1$ and $\Phi < 0 \implies a = 0$
$\lambda_i(v) > 0 \implies$ allocation constant
$\alpha_{i,i-1}(v) > 0 \implies$ utility of i,i-1 equal at v

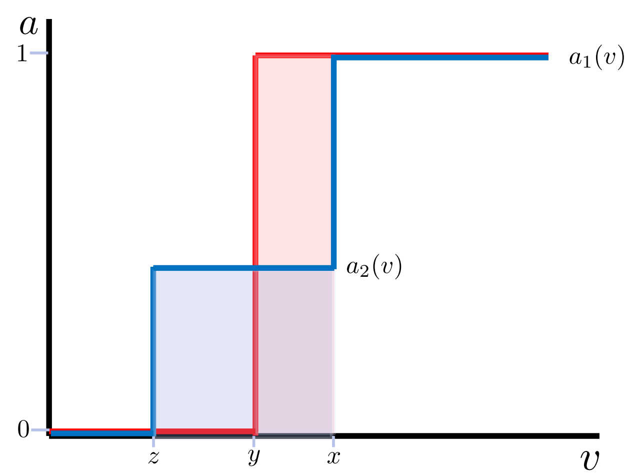
# Implications for the Primal

VIA COMPLEMENTARY SLACKNESS

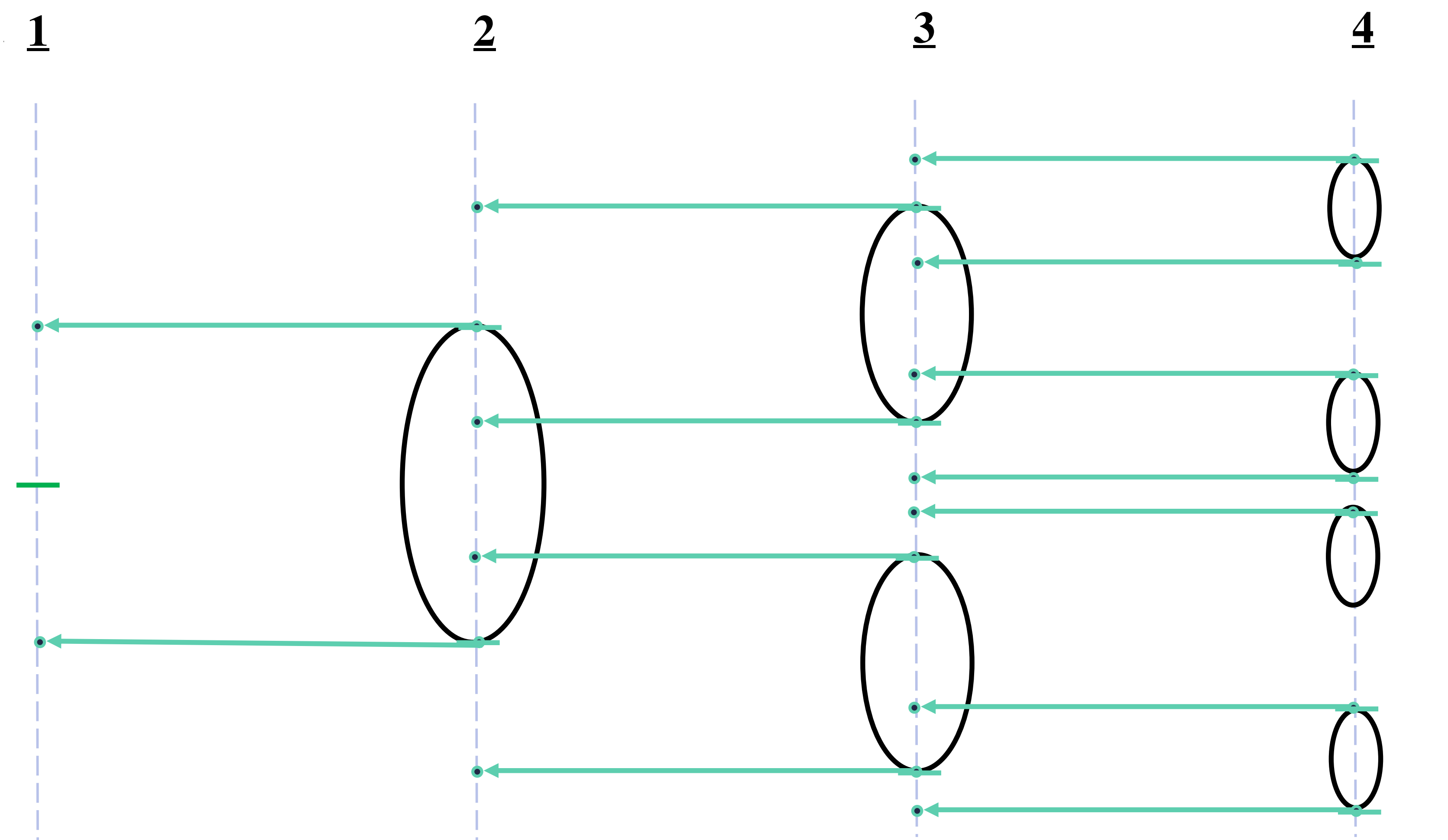
# Splitting the allocation

$\Phi > 0 \Longrightarrow a = 1$ and $\Phi < 0 \Longrightarrow a = 0$
$\lambda_i(v) > 0 \Longrightarrow$ allocation constant
$\alpha_{i,i-1}(v) > 0 \Longrightarrow$ utility of i,i-1 equal at v

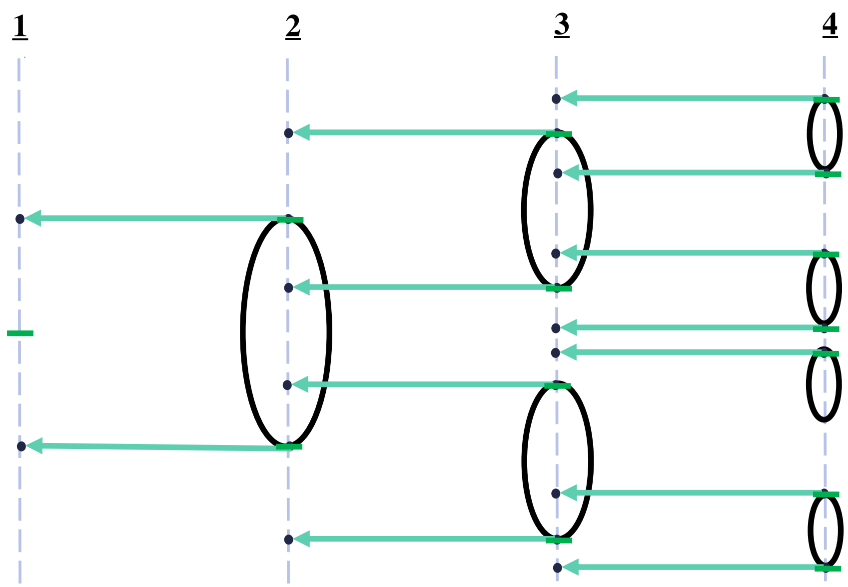


# FedEx Worst Case



# FedEx Menu Complexity




- Exponentially many prices for day i ($2^{i-1}$)
- Exponentially many prices total ($2^n$-1) **[Fiat G. Karlin Koutsoupias '16]**
- Proven to be tight. **[Saxena Schvartzman Weinberg '18]**

[Devanur Weinberg ’17]

# The Budgets Setting


value $v$ =
how much the item is worth

**Budget options**

$B_1$

$B_2$

***(v,B) ~ F***

$B_3$

budget $B$
= how much they can afford

•
•
•

**Result:** At most $3 \cdot 2^{n-1}-1$ prices.

$B_n$

# Partially-Ordered Items

[DEVANUR GOLDNER SAXENA SCHVARTZMAN WEINBERG 2018]

[Devanur G. Saxena Schvartzman Weinberg ’18]

# The Partially-Ordered Setting

interest *G*

= service or set of goods desired

$(v,G) \sim F$

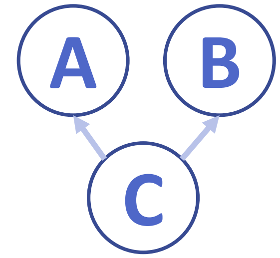
value *v* =

how much getting their interest is worth

**Service options**

# An Optimal Primal/Dual Pair


minimize $_{\lambda,\alpha}$ maximize $_{\text{feasible } a}$




$$\sum_G \int_0^H f_G(v) a_G(v)\, \Phi_G(v) dv$$

Can't change $\lambda$,$\alpha$ to further minimize.

## Complementary Slackness:

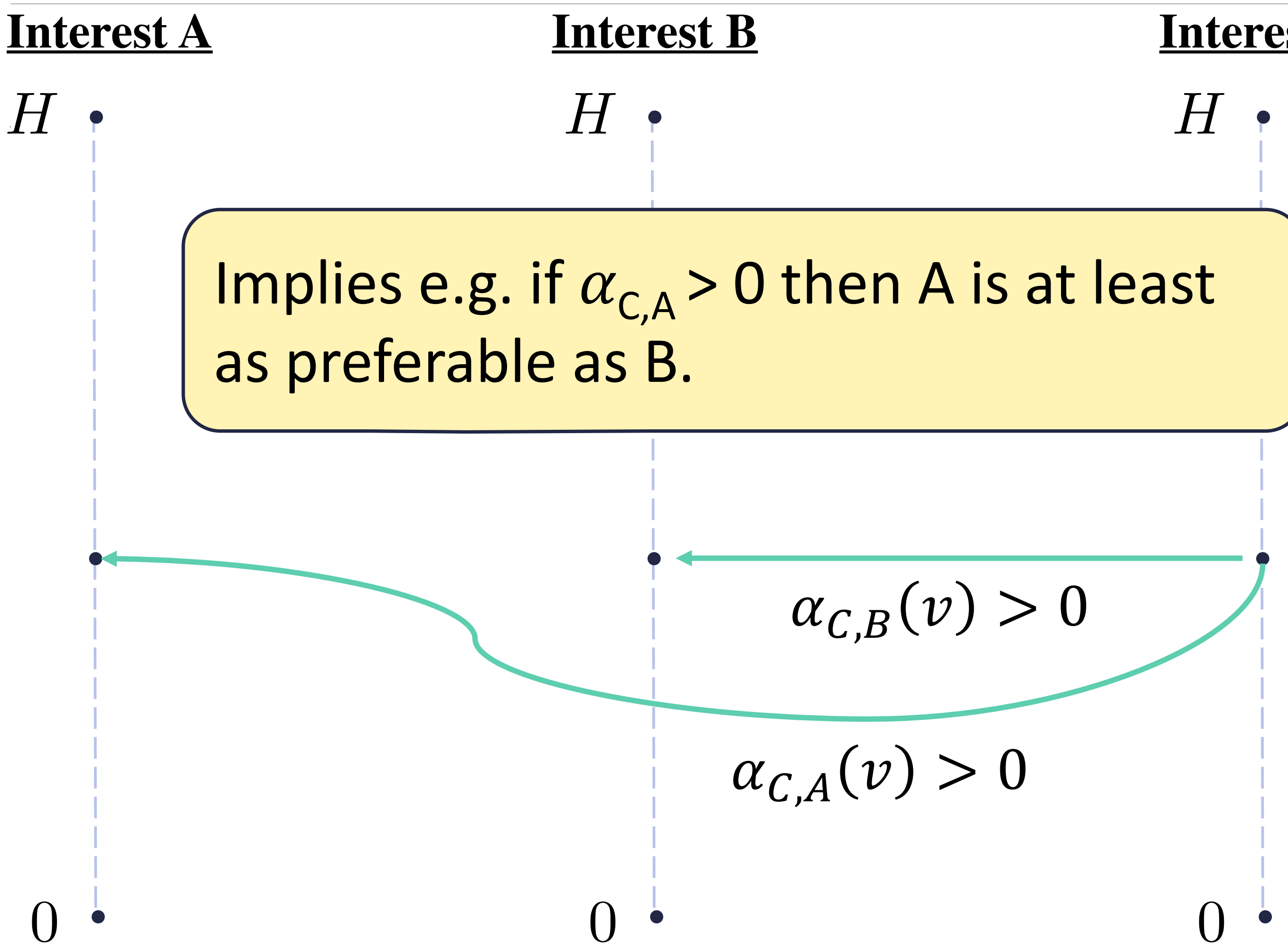
**Constraint** is tight (= 0) or **dual variable** is 0.

For all $G' \in N^+(G)$

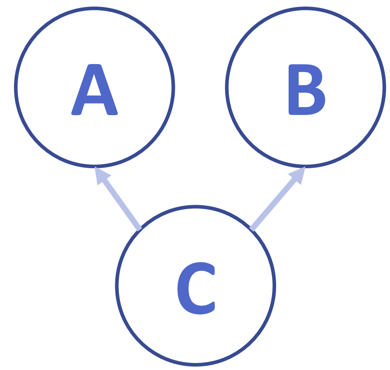
Dual variables

| | | |
|---|---|---|
| Report G over G' | $\int_0^v a_G(x)dx - \int_0^v a_{G'}(x)dx \geq 0$ | $\alpha_{G,G'}(v)$ |
| Report v over v' | $a'_G(v) \geq 0$ | $\lambda_G(v)$ |

# Dual Variables and Virtual Values




Implies e.g. if $\alpha_{C,A} > 0$ then A is at least as preferable as B.

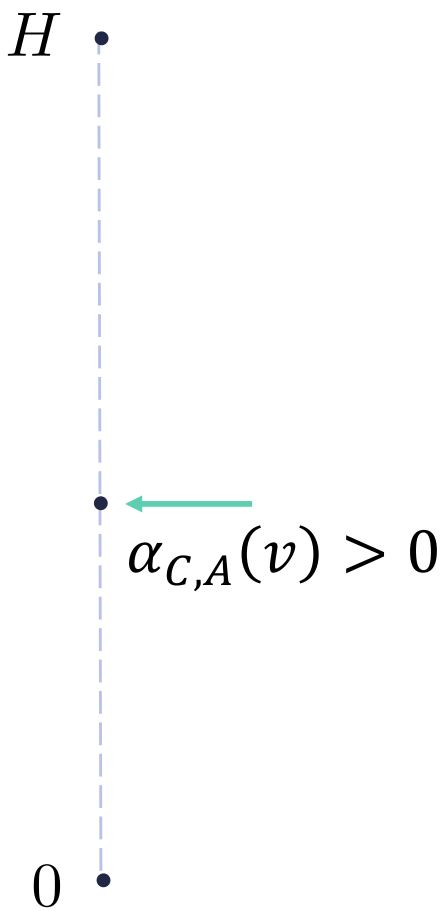



**Complementary Slackness:**

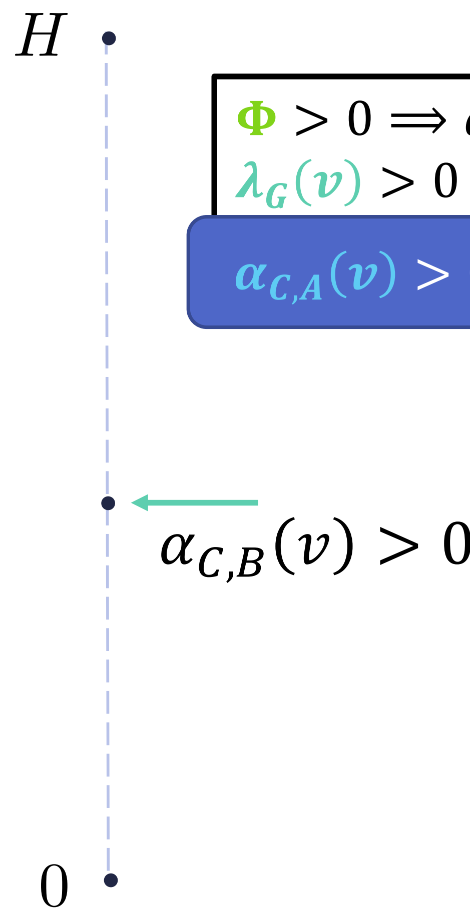
Utility of G and G' are equal where $\alpha_{G,G'} > 0$.

# Dual Variables and Virtual Values

**Interest A**

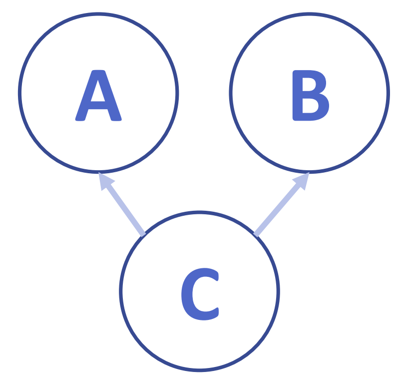



**Interest B**



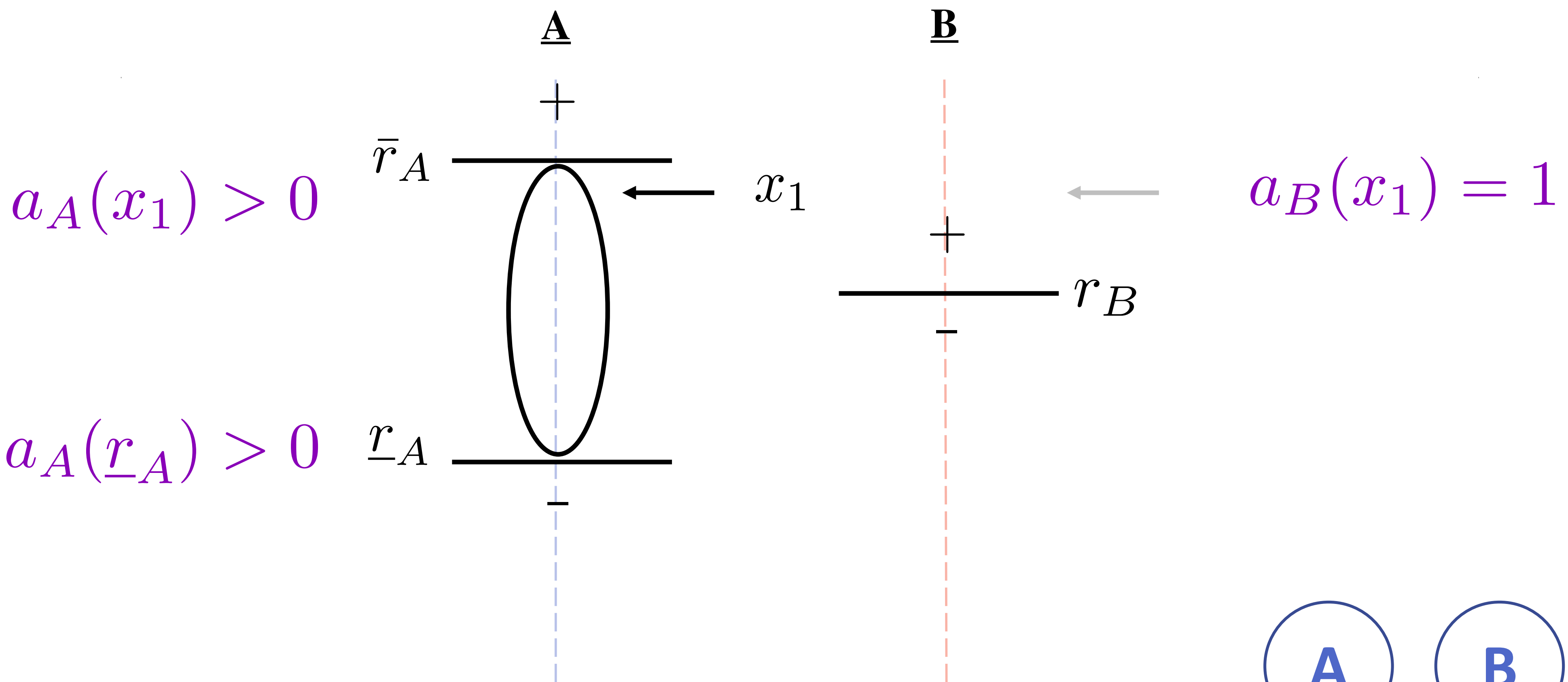

$\Phi > 0 \Longrightarrow a = 1$ and $\Phi < 0 \Longrightarrow a = 0$
$\lambda_G(v) > 0 \Longrightarrow$ allocation constant

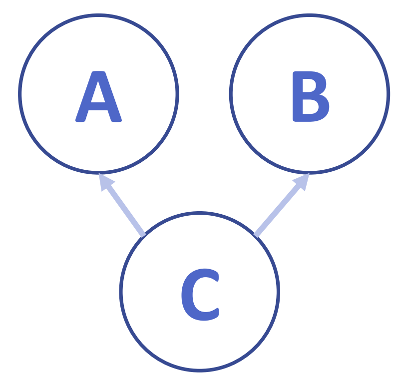
$\alpha_{C,A}(v) > 0 \Longrightarrow$ A is preferable to B at v



# Menu Complexity Lower Bound

# Key Idea for the Lower Bound




$\Phi > 0 \Longrightarrow a = 1$ and $\Phi < 0 \Longrightarrow a = 0$
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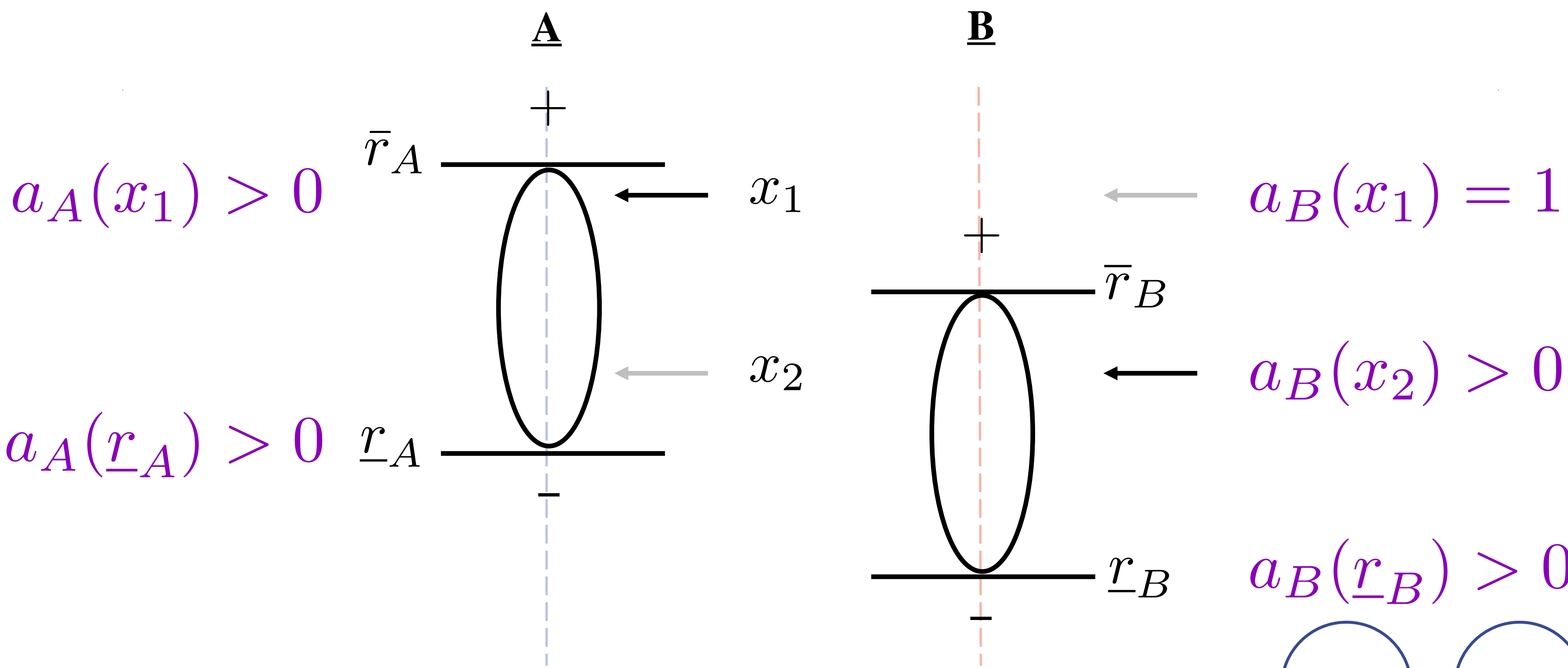


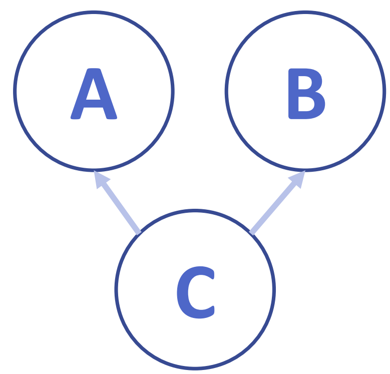


$\Phi > 0 \implies a = 1$ and $\Phi < 0 \implies a = 0$
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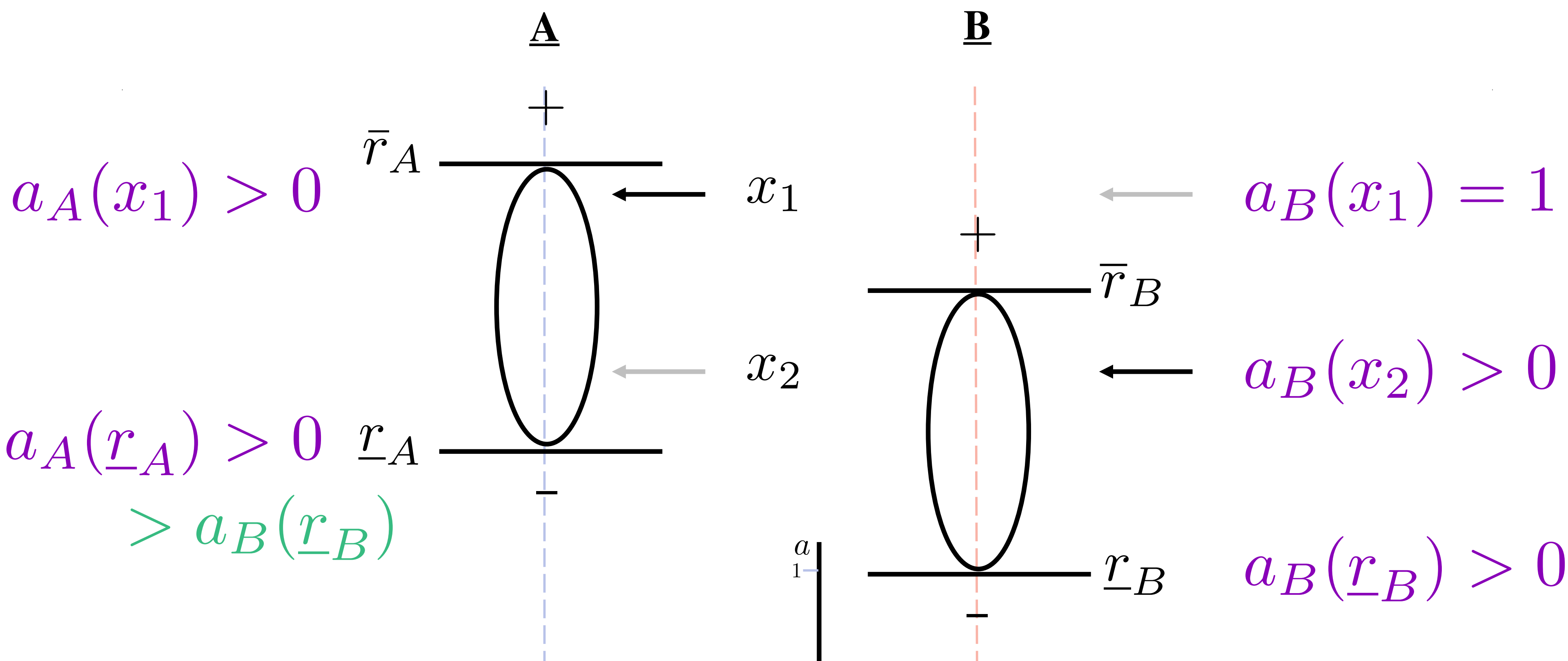
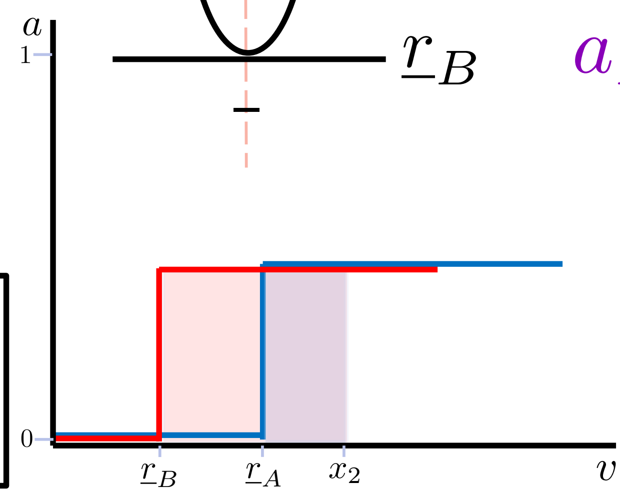
# Key Idea for the Lower Bound




$\Phi > 0 \Longrightarrow a = 1$ and $\Phi < 0 \Longrightarrow a = 0$

$\lambda_G(v) > 0 \Longrightarrow$ allocation constant

$\alpha_{C,A}(v) > 0 \Longrightarrow$ A is preferable to B at v

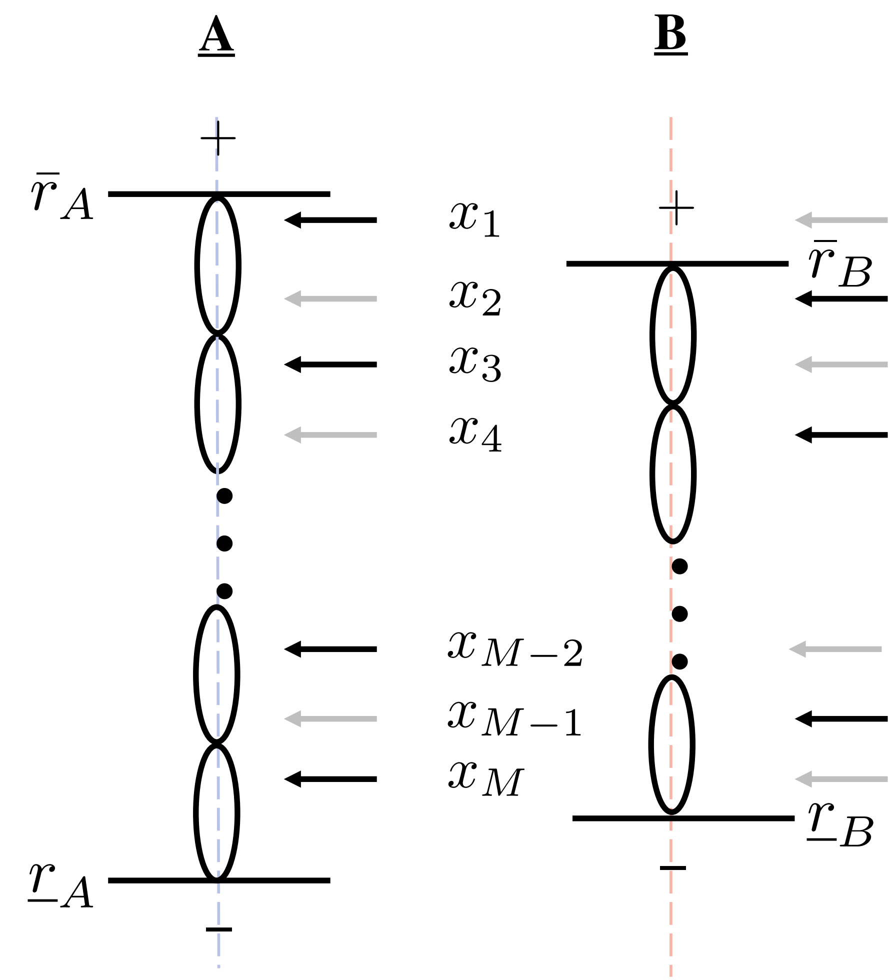


# Lower Bound

$\Phi > 0 \Rightarrow a = 1$ and $\Phi < 0 \Rightarrow a = 0$
$\lambda_G(v) > 0 \Rightarrow$ allocation constant
$\alpha_{C,A}(v) > 0 \Rightarrow$ A is preferable to B at v

**For any M:**

$a_A(x_1) > 0$
$> a_B(x_2)$
$a_A(x_3) > 0$
$> a_B(x_4)$
⋮
$a_A(\underline{r}_A) > 0$



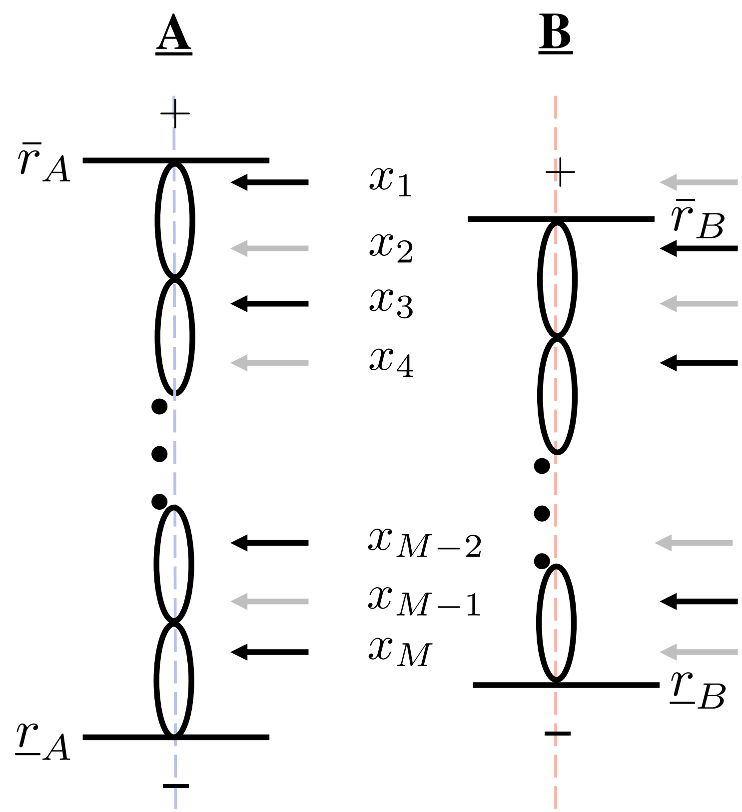

$a_B(x_1) = 1$
$a_B(x_2) > 0$
$> a_A(x_3)$
$a_B(x_4) > 0$
$> a_A(x_5)$
⋮
$a_B(\underline{r}_B) > 0$
$> a_A(\underline{r}_A)$

**> M different options are presented to the buyer.**

# Master Theorem (Informal)

For any dual that is given only by **signs** and **nonnegative variables** (ironed intervals + $\alpha$ flow), there exists a distribution that causes this dual.

**Corollary:**
The “bad dual” exists.



# Menu Complexity Upper Bound

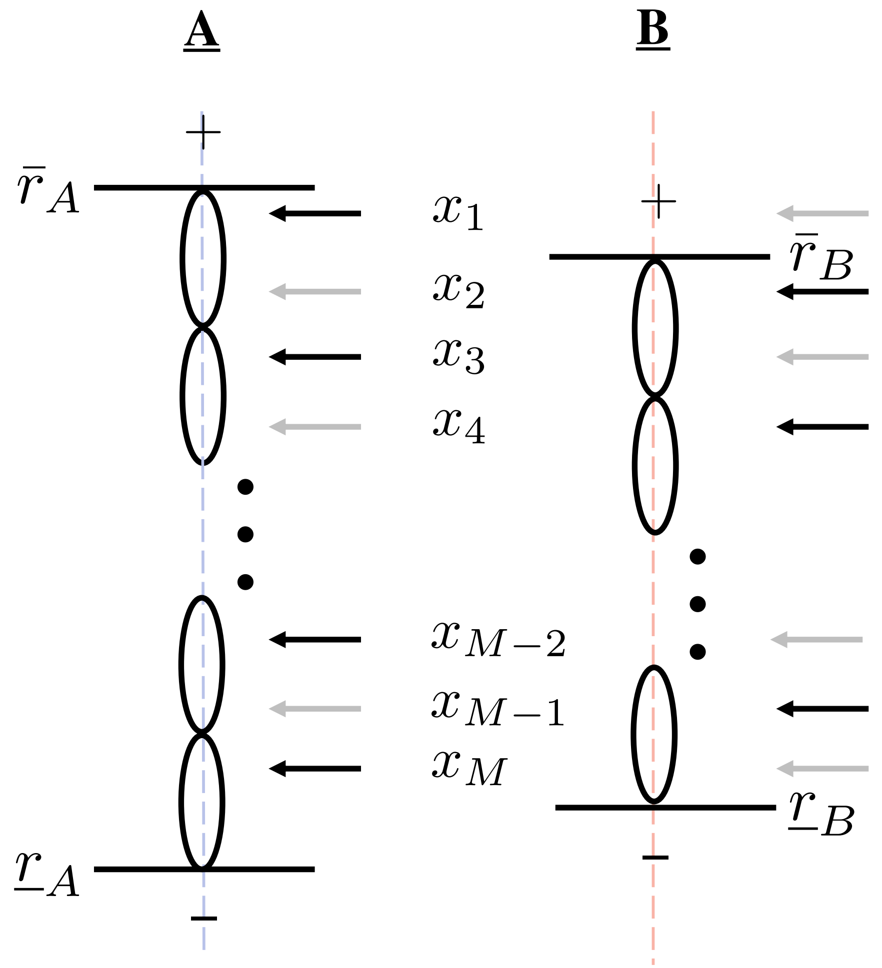
# Upper bound

A **chain** is a sequence of **overlapping ironed intervals** with $\alpha > 0$ at specific points.

If there are M such intervals, the menu size is at most 2M – finite.

If there are **infinitely** many intervals, they're bounded and monotone, so they **converge** to a point that has virtual value 0 and is un-ironed for both A and B – menu size 1.

**Always finite!**



# Multi-Unit Pricing Lower Bound

[Devanur Haghpanah Psomas ’17]

# The Multi-Unit Pricing Setting

value $v$ =

how much each item is worth

**# item options**

1

2

3

•
•
•

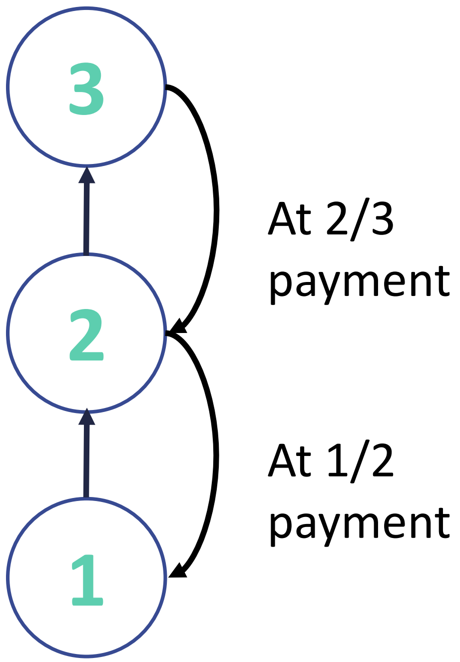
$n$

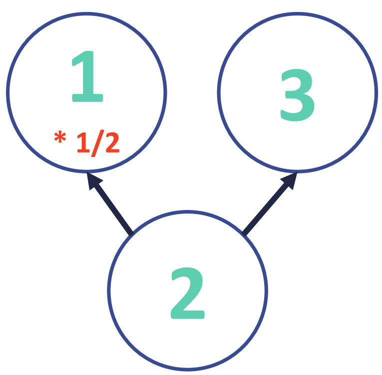
***(v,d) ~ F***

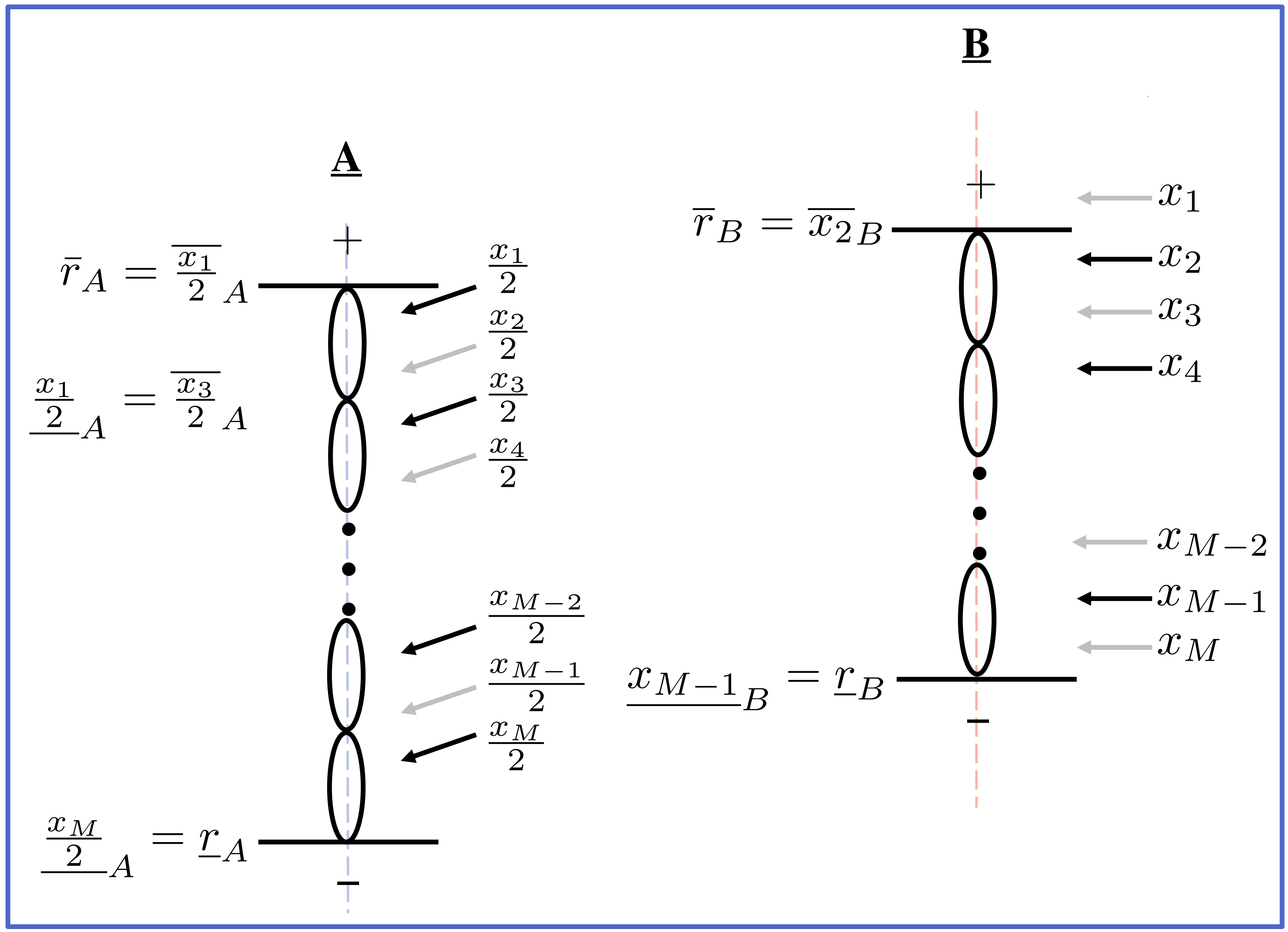
demand $d$

= how many units they want

# Extension to MUP



# Summary

# The Settings

Each buyer has a **most-preferred-outcome** (e.g. 3-day shipping).

The outcomes are **structured** such that a buyer's value for this outcome tells you his value for all outcomes.

Properties:

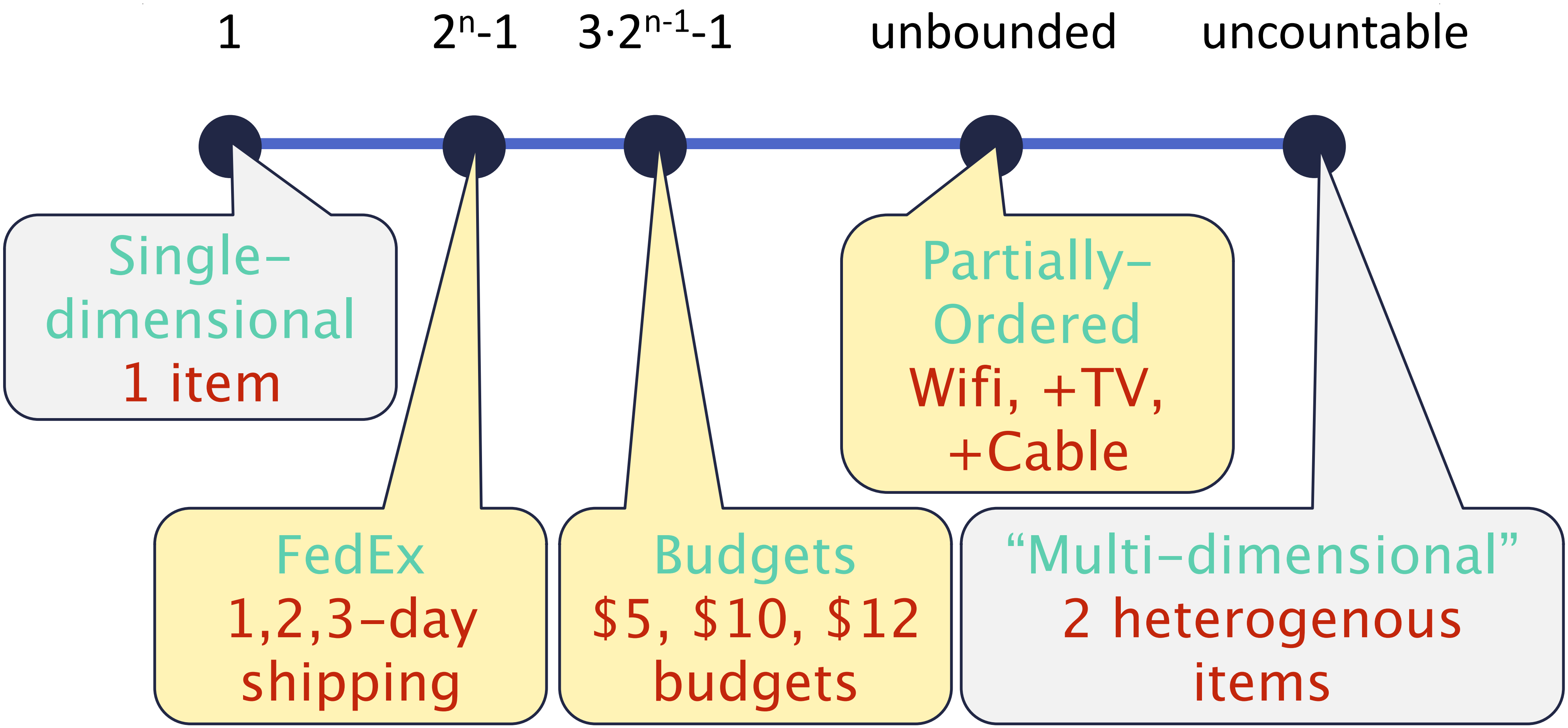
- **Collapsible allocation rule:** degree of happiness
- **Reduced IC constraints:** specified by structure
- **Single-dimensional perks:** payment identity, etc

# The Methods

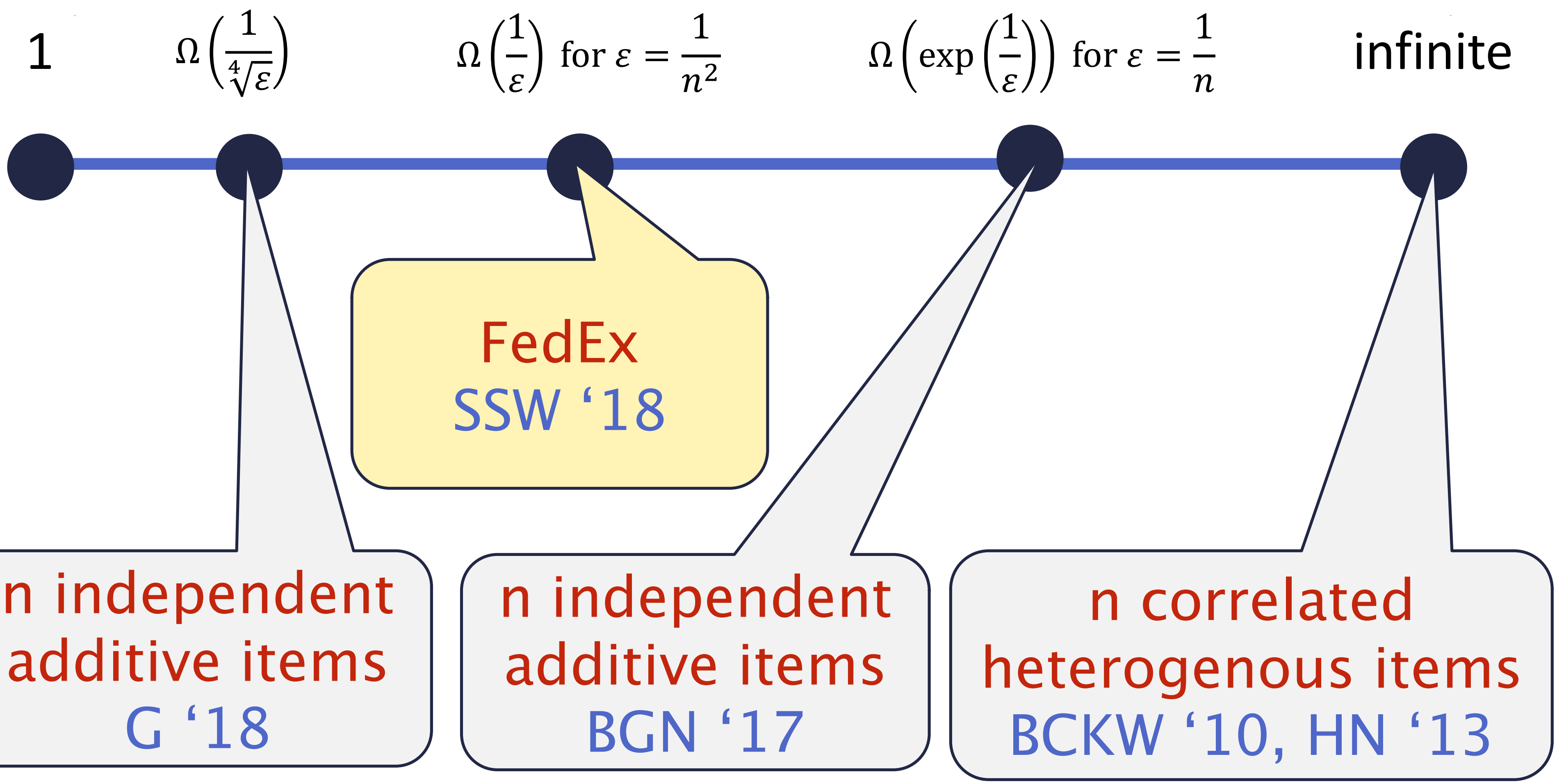
**Revenue Curves:**
- Exactly where complexity grows or “splits”
- Limits of splitting
- Approximation via polygons

**Complementary Slackness conditions:**
- Where are certain outcomes preferred?
- Where must the allocation be positive?
- Where must the allocation be distinct, forcing different menu options?
- What are the limits to this?

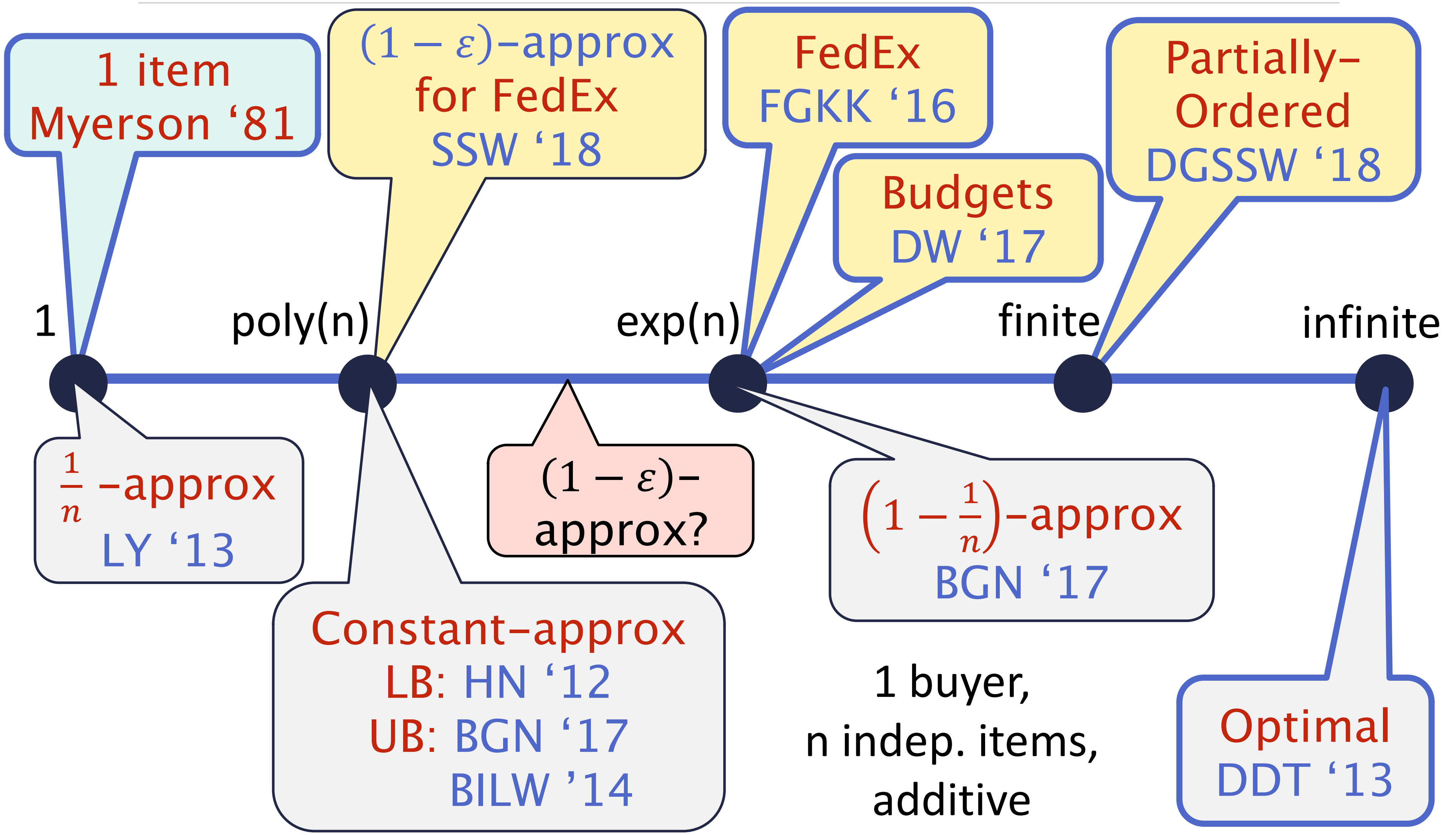
# Optimal Menu Complexity Spectrum



# Lower Bounds for $(1-\varepsilon)$-approximations

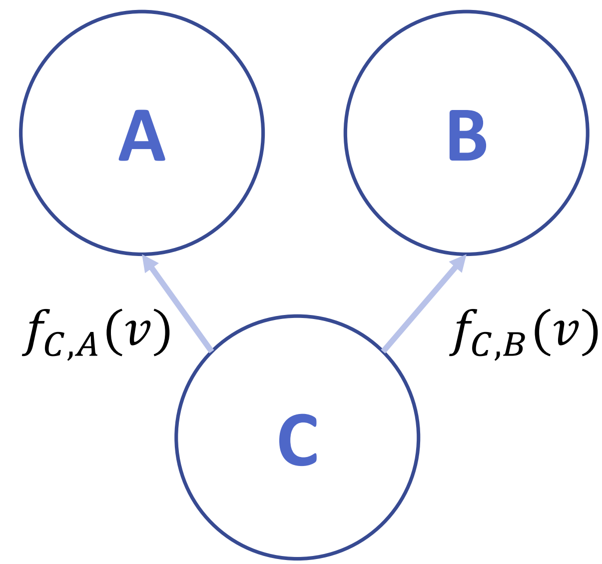


# Multi-Dimensional Menu Complexity for *n* Items



# Key Open Problems

- Other settings with more complex IC links?
- Lower bounds in terms of $\varepsilon$?
- Constant-factor approximations?
- Multiple bidders?
- Filling out the questions asked in Yannai's talk in this setting.



# Thank you!