DS 320 Algorithms for Data Science **Lecture #1**
Spring 2022 Prof. Kira Goldner

Covered in introduction slides:

- Course policies (also in syllabus).
- What to expect in this class (also in FAQ).
- Sample of content we'll cover.

Runtime Review

We will analyze the runtime of the following algorithm.

Algorithm 1 FindMinIndex($B[t+1, n]$).

```
Let MinIndex = t + 1.
for i = t + 1 to n do
    if B[i] < B[MinIndex] then
        MinIndex = i.
    end if
end for
return MinIndex.
```

When we analyze runtime, we'll do an informal accounting. We'll count basic operations (algebra, array assignment, etc) as constant time. This isn't quite right—for example, multiplication of large numbers should scale with the bit complexity—but is a good approximation for us. We will analyze runtime by counting these operations.

Each of the following lines is a unit (constant-time) operation:

- **Let** MinIndex $= t+1$.
- **if** $B[i] < B[\text{MinIndex}]$ **then**
- MinIndex $= i$.

The for-loop runs $n - t$ times (notice that both n and t are variables as they are in our input). Thus the runtime of this algorithm is $O(n - t)$.

Asymptotic Notation

Definition 1 (Upper bound $O(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in O(g(n))$ if there exist constants c_1, c_2 such that for all $n \geq c_1$,

$$f(n) \leq c_2 g(n).$$

We'll often write $f(n) = O(g(n))$ because we are sloppy.

Translation: For large n (at least some c_1), the function $g(n)$ dominates $f(n)$ up to a constant factor.

Definition 2 (Lower bound $\Omega(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in \Omega(g(n))$ if there exist constants c_1, c_2 such that for all $n \geq c_1$,

$$f(n) \geq c_2 g(n).$$

Definition 3 (Tight bound $\Theta(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in \Theta(g(n))$ if $f \in O(g(n))$ and $f \in \Omega(g(n))$.

Exercise: True or False?

$f(n)$	$g(n)$	$O(g(n))$	$\Omega(g(n))$	$\Theta(g(n))$
$10^6 n^3 + 2n^2 - n + 10$	n^3	T	T	T
$\sqrt{n} + \log n$	$\sqrt{n}$	T	T	T
$n(\log n + \sqrt{n})$	$\sqrt{n}$	F	T	F
n	n^2	T	F	F

Example solution: Let $f(n) = 10^6 n^3 + 2n^2 - n + 10$. For $c_2 = (10^6 + 12)$, $10^6 n^3 + 2n^2 - n + 10 \leq c_2 n^3$ for all $n \geq 1$, hence it is true that $f(n) = O(n^3)$. For $c_2 = 1$, $10^6 n^3 + 2n^2 - n + 10 \leq c_2 n^3$, hence it is true that it is $f(n) = \Omega(n^3)$. Since $f(n) = O(n^3)$ and $f(n) = \Omega(n^3)$, then $f(n) = \Theta(n^3)$ as well.

Definition 4 (Strict upper bound $o(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in o(g(n))$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0,$$

or equivalently, for *any* constant $c_2 > 0$, there exists a constant c_1 such that for all $n \geq c_1$,

$$f(n) < c_2 g(n).$$

Definition 5 (Strict lower bound $\omega(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in \omega(g(n))$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty,$$

or equivalently, for *any* constant $c_2 > 0$, there exists a constant c_1 such that for all $n \geq c_1$,

$$f(n) > c_2 g(n).$$

Asymptotic Properties

- Multiplication by a constant:

  If $f(n) = O(g(n))$ then for any $c > 0$, $c \cdot f(n) = O(g(n))$.

- Transitivity:

  If $f(n) = O(h(n))$ and $h(n) = O(g(n))$ then $f(n) = O(g(n))$.

- Symmetry:

  If $f(n) = O(g(n))$ then $g(n) = \Omega(f(n))$.

  If $f(n) = \Theta(g(n))$ then $g(n) = \Theta(f(n))$.

- Dominant Terms:

  If $f(n) = O(g(n))$ and $d(n) = O(e(n))$ then $f(n) + d(n) = O(\max\{g(n), e(n)\})$. It's fine to write this as $O(g(n) + e(n))$.

Common Functions

- Polynomials: $a_0 + a_1 n + \cdots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$.

- Polynomial time: Running time is $O(n^d)$ for some constant d independent of the input size n.

- Logarithms: $\log_a n = \Theta(\log_b n)$ for all constants $a, b > 0$. This means we can avoid specifying the base of the logarithm.

  For every $x > 0$, $\log n = o(n^x)$. Hence log grows slower than every polynomial.

- Exponentials: For all $r > 1$ and all $d > 0$, $n^d = o(r^n)$. Every polynomial grows slower than every exponential

- Factorial: By Sterling's formula, factorials grow faster than every exponential:

$$n! = (\sqrt{2\pi n}) \left(\frac{n}{e}\right)^n (1 + o(1)) = 2^{\Theta(n \log n)}.$$