# **Dynamic Programming IV: Shortest Paths**

# The Problem

Given a graph G = (V, E) with any edge weights  $w_e$  for all  $e \in E$ , a source s, output the shortest s - v path for every  $v \in V$ .

## Naïve Approach

Let N(v) be the neighbors of v.

- Subproblem: Let OPT(v) denote the length of the shortest path from s to v.
- Recurrence:  $OPT(v) = \min_{u \in N(v)} OPT(u) + w_{(u,v)}$ .
- Base Case: OPT(s) = 0.

What's wrong with this approach? List any issues you see:

# The Bellman-Ford Algorithm

Main Idea: Impose a measure of progress—parametrize the subproblems.

More specifically:

- Consider a shortest path from  $s \to u \to v \to t$  with k edges
- $s \to u \to v$  is a shortest  $s \to v$  path with k-1 edges

Measure of progress: the number of edges in path. First compute all shortest paths with  $\leq 1$  edges. Then shortest paths with  $\leq 2$  edges. And so on, until we compute shortest paths of length  $\leq n-1$ .

#### **Structural Observation:**

Proof.

Formal Description State Your Subproblem:

State Your Recurrence:

**Prove Your Recurrence:** 

State Your Base Cases:

#### **Present Your Algorithm:**

# **Runtime:**

#### • Size of table:

• Time to fill in table:

(How many table lookups?)

Total time:

#### **Better Runtime Analysis**

- How many row updates?
- How many table lookups per row update?
- Runtime =

## **Detecting Negative Cycles**

Idea: Add an extra column (Column n) to the memo table.

**Claim 1.** There is a negative cycle in the graph if and only if Column  $n \neq$  Column n-1.

*Proof.* ( $\Leftarrow$ ) Suppose the graph has no negative cycles. Then by the Structural Observation,

 $(\Rightarrow)$  Suppose there is a negative cycle in the graph.

To detect negative cycles:

- Run Bellman-Ford, add an nth column.
- If Column n = Column n 1, return "No Negative Cycles"
- Otherwise, return that there's a negative cycle.

# Space-Efficient Bellman-Ford

How much space does Bellman-Ford use?

## **Observation:**

## Improved algorithm:

Total space usage: