DS 320 Algorithms for Data Science	**Lecture #19 Worksheet**
Spring 2022	Prof. Kira Goldner

Linear Programming II: Duality

Example 1: Grain Nutrients

Suppose BU has hired you to optimize nutrition for campus dining. There are two possible grains they can offer, grain 1 and grain 2, and each contains the macronutrients found in the table below, plus cost per kg for each of the grains.

Macros	Starch	Proteins	Vitamins	Cost ($/kg)
Grain 1	5	4	2	0.6
Grain 2	7	2	1	0.35

The nutrition requirement per day of starch, proteins, and vitamins is 8, 15, and 3 respectively. Determine how much of each grain to buy such that BU spends as little but meets its nutrition requirements.

Decision variables:

Objective:

Constraints:

Example 2: Transportation

You're working for a company that's producing widgets among two different factories and selling them from three different centers. Each month, widgets need to be transported from the factories to the centers. Below are the transportation costs from each factory to each center, along with the monthly supply and demand for each factory and center respectively. Determine how to route the widgets in a way that minimizes transportation costs.

Transit Cost	Center 1	Center 2	Center 3
Factory 1	5	5	3
Factory 2	6	4	1

- The supply per factory is 6 and 9 respectively.
- The demand per center is 8, 5, and 2 respectively.

Decision variables:

Objective:

Constraints:

Converting to Normal Form

The "Normal Form" of a Linear Program looks like:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned}$$

Our Transportation problem had the LP:

$$\begin{aligned} \min \quad & 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 1x_{23} & \\ \text{subject to} \quad & x_{11} + x_{12} + x_{13} = 6 & \text{(Factor 1 supply)} \\ & x_{21} + x_{22} + x_{23} = 9 & \text{(Factor 2 supply)} \\ & x_{11} + x_{21} = 8 & \text{(Center 1 demand)} \\ & x_{12} + x_{22} = 5 & \text{(Center 2 demand)} \\ & x_{13} + x_{23} = 2 & \text{(Center 3 demand)} \\ & x_{ij} \ge 0 & \text{(non-negativity)} \end{aligned}$$

How can we convert it to normal form—a maximization problem with all less-than-or-equal-to constraints?

First observe that $x_{11} + x_{12} + x_{13} = 6$ is equivalent to having both inequalities

But, we need both to be $\le$ inequalities! We transform them to

The resulting LP in normal form is:

The Dual of a Linear Program

Every linear program has a *dual* linear program. We call the original linear program the *primal*. There are a bunch of amazing properties that come from LP duality.

Going back to our nutrition example, we want to find the dual linear program. A maximization problem's dual is a minimization problem. Here, we have a minimization problem, so the dual will be a maximization problem.

Primal:

$$\begin{aligned} \min \quad & 0.6y_1 + 0.35y_2 & \\ \text{subject to} \quad & 5y_1 + 7y_2 \geq 8 & \text{(starch)} \\ & 4y_1 + 2y_2 \geq 15 & \text{(proteins)} \\ & 2y_1 + 1y_2 \geq 3 & \text{(vitamins)} \\ & y_1, y_2 \geq 0 & \text{(non-negativity)} \end{aligned}$$

Dual:

Sometimes, the dual can even be interpreted as a related problem. You might interpret this problem as follows. You're selling nutrients to the BU population and deciding what to price each macro at. The decision variables x_i will indicate the price per nutrient. The constraints indicate that these prices together cannot exceed the prices for the grains that you're extracting the nutrients from, since that's already the market price. The goal is to maximize your profits from a population that is buying exactly the nutrient diet of 8kg starch, 15kg proteins, and 3kg vitamins.

The following is the normal form for a maximization problem primal and its primal:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \end{aligned}$$

For the above example:

$\mathbf{A} =$ $\qquad$ $\mathbf{b} =$ $\qquad$ $\mathbf{c} =$

Example 3: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges S such that no vertex is covered by more than one edge.

Decision variables:

Linear Program:

Taking the dual of the above primal, we get the following linear program:

What problem is this?