Linear Programming II: Duality

Example 1: Grain Nutrients

Suppose BU has hired you to optimize nutrition for campus dining. There are two possible grains they can offer, grain 1 and grain 2, and each contains the macronutrients found in the table below, plus cost per kg for each of the grains.

Macros	Starch	Proteins	Vitamins	Cost ($/kg)
Grain 1	5	4	2	0.6
Grain 2	7	2	1	0.35

The nutrition requirement per day of starch, proteins, and vitamins is 8, 15, and 3 respectively. Determine how much of each grain to buy such that BU spends as little but meets its nutrition requirements.

Decision variables: amount of grain 1 (y_1) and grain 2 (y_2).

Objective: Minimize cost.

$$\min 0.6y_1 + 0.35y_2$$

Constraints:

$$\begin{aligned} 5y_1 + 7y_2 &\geq 8 && \text{(starch)} \\ 4y_1 + 2y_2 &\geq 15 && \text{(proteins)} \\ 2y_1 + 1y_2 &\geq 3 && \text{(vitamins)} \\ y_1, y_2 &\geq 0 && \text{(non-negativity)} \end{aligned}$$

Example 2: Transportation

You're working for a company that's producing widgets among two different factories and selling them from three different centers. Each month, widgets need to be transported from the factories to the centers. Below are the transportation costs from each factory to each center, along with the monthly supply and demand for each factory and center respectively. Determine how to route the widgets in a way that minimizes transportation costs.

Transit Cost	Center 1	Center 2	Center 3
Factory 1	5	5	3
Factory 2	6	4	1

- The supply per factory is 6 and 9 respectively.

- The demand per center is 8, 5, and 2 respectively.

Decision variables: x_{ij} is the number of widgets transported from factory i to center j.

Objective: Minimize cost.

$$\min \quad 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 1x_{23}$$

Constraints:

$$\begin{aligned}
x_{11} + x_{12} + x_{13} &= 6 && \text{(Factor 1 supply)} \\
x_{21} + x_{22} + x_{23} &= 9 && \text{(Factor 2 supply)} \\
x_{11} + x_{21} &= 8 && \text{(Center 1 demand)} \\
x_{12} + x_{22} &= 5 && \text{(Center 2 demand)} \\
x_{13} + x_{23} &= 2 && \text{(Center 3 demand)} \\
x_{ij} &\geq 0 && \text{(non-negativity)}
\end{aligned}$$

Converting to Normal Form

The "Normal Form" of a Linear Program looks like:

$$\begin{aligned}
\max \quad & \mathbf{c}^T\mathbf{x} \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}
\end{aligned}$$

Our Transportation problem had the LP:

$$\begin{aligned}
\min \quad & 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 1x_{23} \\
\text{subject to} \quad & x_{11} + x_{12} + x_{13} = 6 && \text{(Factor 1 supply)} \\
& x_{21} + x_{22} + x_{23} = 9 && \text{(Factor 2 supply)} \\
& x_{11} + x_{21} = 8 && \text{(Center 1 demand)} \\
& x_{12} + x_{22} = 5 && \text{(Center 2 demand)} \\
& x_{13} + x_{23} = 2 && \text{(Center 3 demand)} \\
& x_{ij} \geq 0 && \text{(non-negativity)}
\end{aligned}$$

How can we convert it to normal form—a maximization problem with all less-than-or-equal-to constraints?

First observe that $x_{11} + x_{12} + x_{13} = 6$ is equivalent to having both inequalities

$$x_{11} + x_{12} + x_{13} \leq 6 \qquad \text{and} \qquad x_{11} + x_{12} + x_{13} \geq 6.$$

But, we need both to be $\leq$ inequalities! We transform them to

$$x_{11} + x_{12} + x_{13} \leq 6 \qquad \text{and} \qquad -x_{11} - x_{12} - x_{13} \leq -6.$$

The resulting LP in normal form is:

$$\begin{array}{rll}
\max & -5x_{11} - 5x_{12} - 3x_{13} - 6x_{21} - 4x_{22} - 1x_{23} & \\
\text{subject to} & x_{11} + x_{12} + x_{13} \leq 6 & \text{(Factor 1 supply)} \\
& -x_{11} - x_{12} - x_{13} \leq -6 & \text{(Factor 1 supply)} \\
& x_{21} + x_{22} + x_{23} \leq 9 & \text{(Factor 2 supply)} \\
& -x_{21} - x_{22} - x_{23} \leq -9 & \text{(Factor 2 supply)} \\
& x_{11} + x_{21} \leq 8 & \text{(Center 1 demand)} \\
& -x_{11} - x_{21} \leq -8 & \text{(Center 1 demand)} \\
& x_{12} + x_{22} \leq 5 & \text{(Center 2 demand)} \\
& -x_{12} - x_{22} \leq -5 & \text{(Center 2 demand)} \\
& x_{13} + x_{23} \leq 2 & \text{(Center 3 demand)} \\
& -x_{13} - x_{23} \leq -2 & \text{(Center 3 demand)} \\
& x_{ij} \geq 0 & \text{(non-negativity)}
\end{array}$$

The Dual of a Linear Program

Every linear program has a *dual* linear program. We call the original linear program the *primal*. There are a bunch of amazing properties that come from LP duality.

Going back to our nutrition example, we want to find the dual linear program. A maximization problem's dual is a minimization problem. Here, we have a minimization problem, so the dual will be a maximization problem.

Primal:

$$\begin{array}{rlll}
\min & 0.6y_1 + 0.35y_2 & & \\
\text{subject to} & 5y_1 + 7y_2 \geq 8 & \text{(starch)} & (x_1) \\
& 4y_1 + 2y_2 \geq 15 & \text{(proteins)} & (x_2) \\
& 2y_1 + 1y_2 \geq 3 & \text{(vitamins)} & (x_3) \\
& y_1, y_2 \geq 0 & \text{(non-negativity)} &
\end{array}$$

Dual:

$$\begin{array}{rlll}
\max & 8x_1 + 15x_2 + 3x_3 & & \\
\text{subject to} & 5x_1 + 4x_2 + 2x_3 \leq 0.6 & \text{(grain 1)} & (y_1) \\
& 7x_1 + 2x_2 + 1x_3 \leq 0.35 & \text{(grain 2)} & (y_2) \\
& x_1, x_2, x_3 \geq 0 & \text{(non-negativity)} &
\end{array}$$

To take the dual: Label each primal constraint with a new dual variable. In our new linear program, each dual constraint will correspond to a primal variable. For the left-hand side, count up the appearances of this constraint's primal variable (e.g., x_1) in each of the primal constraints and multiply them by the dual variable for those constraints. That is, if x_1 appears 5 times ($5x_1$) in constraint for y_1, then add $5y_1$ to x_1's constraint. Don't forget to include its appearance in the primal's objective function, but this will be the right-hand side of the constraint. Finally, the dual objective function is given by the right-hand side coefficients and their correspondence to the dual variables via the constraints in the primal. (See above).

Sometimes, the dual can even be interpreted as a related problem. You might interpret this problem as follows. You're selling nutrients to the BU population and deciding what to price each macro at. The decision variables x_i will indicate the price per nutrient. The constraints indicate that these prices together cannot exceed the prices for the grains that you're extracting the nutrients from, since that's already the market price. The goal is to maximize your profits from a population that is buying exactly the nutrient diet of 8kg starch, 15kg proteins, and 3kg vitamins.

The following is the normal form for a maximization problem primal and its primal:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \end{aligned}$$

For the above example:

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 \\ 7 & 2 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.6 \\ 0.35 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 8 \\ 15 \\ 3 \end{bmatrix}$$

Example 3: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges S such that no vertex is covered by more than one edge.

Decision variables: x_e indicating whether edge e is in the matching.

Primal Linear Program:

$$\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e : v \in e} x_e \le 1 && \forall v \quad \text{(vertex matched at most once)} \quad (y_v) \\ & x_e \ge 0 && \forall e \quad \text{(non-negativity)} \end{aligned}$$