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| **DS 320 Algorithms for Data Science** | **Lecture #20** |
| **Spring 2022** | Prof. Kira Goldner |

# Linear Programming III: More Duality

To take the dual: Label each primal constraint with a new dual variable. In our new linear program, each dual constraint will correspond to a primal variable. For the left-hand side, count up the appearances of this constraint's primal variable (e.g., $x_1$) in each of the primal constraints and multiply them by the dual variable for those constraints. That is, if $x_1$ appears 5 times ($5x_1$) in constraint for $y_1$, then add $5y_1$ to $x_1$'s constraint. Don't forget to include its appearance in the primal's objective function, but this will be the right-hand side of the constraint. Finally, the dual objective function is given by the right-hand side coefficients and their correspondence to the dual variables via the constraints in the primal. (See above).

The following is the normal form for a maximization problem primal and its primal:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \end{aligned}$$

### Example 3: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges $S$ such that no vertex is covered by more than one edge.

Decision variables: $x_e$ indicating whether edge $e$ is in the matching.

Primal Linear Program:

$$\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e: v \in e} x_e \leq 1 && \forall v \quad \text{(vertex matched at most once)} \quad (y_v) \\ & x_e \geq 0 && \forall e \quad \text{(non-negativity)} \end{aligned}$$

Taking the dual of the above primal, we get the following linear program:

$$\begin{aligned} \min \quad & \sum_{v \in V} y_v \\ \text{subject to} \quad & \sum_{v \in e} y_v \geq 1 && \forall e \quad \text{(edge covered)} \quad (x_e) \\ & y_v \geq 0 && \forall v \quad \text{(non-negativity)} \end{aligned}$$

What problem is this? (Fractional) Vertex Cover!

## Weak Duality

**Theorem 1** (Weak Duality)**.** *If* $\mathbf{x}$ *is feasible in (P) and* $\mathbf{y}$ *is feasible in (D) then* $\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}$*.*

*Proof.*

$$\mathbf{c}^T\mathbf{x} \quad \overset{1}{\le} \quad (\mathbf{A}^T\mathbf{y})\mathbf{x} \quad = \quad \mathbf{y}^T\mathbf{A}\mathbf{x} \quad \overset{2}{\le} \quad \mathbf{y}^T\mathbf{b} \quad = \quad \mathbf{b}^T\mathbf{y}.$$

Where (1) follows by the dual constraints $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$ and (2) follows by the primal constraints $\mathbf{A}\mathbf{x} \le \mathbf{b}$. □

This theorem says that *any* feasible solution to the primal is a *lower bound* to *any* feasible solution to the dual, and likewise, any feasible solution to the dual is an *upper bound* to the primal.

That is, fractional vertex cover gives an upper bound on how large the (fractional) maximum matching can be, and likewise, fractional maximum matching gives a lower bound on how small the minimum (fractional) vertex cover can be.

**Theorem 2** (Strong Duality)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if* $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$*.*

The *if* direction is easy to see: we know that the dual gives an upper bound on the primal, so if these objectives are equal, then the primal objective that we are trying to maximize could not possible get any larger, as it's always *at most* the dual's objective. This is *as tight as possible.*

The *only if* direction is harder to prove, and we'll skip it for now.

## Complementary Slackness

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \end{aligned}$$

or equivalently

$$\sum_i a_{ji}x_i \le b_j \quad \forall j \quad (y_j) \qquad\qquad \sum_i a_{ij}y_i \ge c_i \quad \forall i \quad (x_i)$$

Then *complementary slackness* says we must have at least one of

$$\sum_i a_{ji}x_i = b_j \quad \text{or} \quad y_j = 0 \quad (1) \qquad\qquad \sum_i a_{ij}y_i = c_i \quad \text{or} \quad x_i = 0 \quad (2)$$

holding for our solution to be optimal.

**Theorem 3** (Complementary Slackness)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if the above complementary slackness conditions (1) and (2) hold.*

*Proof.* ($\Rightarrow$) According to complementary slackness, by rearranging our constraint, either $\sum_i a_{ji}x_i - b_j = 0$ or $y_j = 0$. This ensures that the multiplied quantity $\left(\sum_i a_{ji}x_i - b_j\right) y_j = 0$, as *one* of the two terms on the left-hand side must be 0. Then multiplying out and rearranging gives that

$y_j \sum_i a_{ji} x_i = y_j b_j$. This process with all rows gives the equality from complementary slackness that $\mathbf{y}^T \mathbf{A} \mathbf{x} = \mathbf{y}^T \mathbf{b}$.

Similarly, using the condition that $\sum_i a_{ij} y_i = c_i$ or $x_i = 0$ gives that $\mathbf{c}^T \mathbf{x} = (\mathbf{A}^T \mathbf{y}) \mathbf{x}$.

Then following our inequalities in the proof of weak duality, they now all hold with equality, so by Strong Duality, $(x, y)$ are optimal solutions to the primal and dual.

$$\mathbf{c}^T \mathbf{x} \quad = \quad (\mathbf{A}^T \mathbf{y}) \mathbf{x} \quad = \quad \mathbf{y}^T \mathbf{A} \mathbf{x} \quad = \quad \mathbf{y}^T \mathbf{b} \quad = \quad \mathbf{b}^T \mathbf{y}.$$

($\Leftarrow$) Similarly, if Strong Duality holds, the above inequalities hold with equality, in which case it must be that $y_j \sum_i a_{ji} x_i = y_j b_j$ for all $j$ and $\sum_i a_{ij} y_i x_i = c_i x_i$ for all $i$, and hence that either $\sum_i a_{ji} x_i - b_j = 0$ or $y_j = 0$ for all $j$ and that either $\sum_i a_{ij} y_i = c_i$ or $x_i = 0$ for all $i$. $\square$