DS 320 Algorithms for Data Science	**Lecture #1**
Spring 2023	Prof. Kira Goldner

Covered in introduction slides:

- Course policies (also in syllabus).
- What to expect in this class (also in FAQ).
- Sample of content we'll cover.

Runtime Review

When we analyze runtime, we'll do an informal accounting. We'll count basic operations (algebra, array assignment, etc) as constant time.[1]

We will analyze the runtime of the following algorithm:

Algorithm 1 FindMinIndex($B[t+1, n]$).

```
Let MinIndex = t + 1.
for i = t + 1 to n do
    if B[i] < B[MinIndex] then
        MinIndex = i.
    end if
end for
return MinIndex.
```

Each of the following lines is a unit (constant-time) operation:

- **Let** MinIndex $= t + 1$.
- **if** $B[i] < B[\text{MinIndex}]$ **then**
- MinIndex $= i$.

The for-loop runs $n - t$ times (notice that both n and t are variables as they are in our input). Thus the runtime of this algorithm is $O(n - t)$.

Asymptotic Notation

Definition 1 (Upper bound $O(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in O(g(n))$ if there exist ($\exists$) constants c_1, c_2 such that for all (s.t. $\forall$) $n \geq c_1$,

$$f(n) \leq c_2 g(n).$$

We'll often write $f(n) = O(g(n))$ because we are sloppy.

[1] This isn't quite right—for example, multiplication of large numbers should scale with the bit complexity—but is a good approximation for us. We will analyze runtime by counting these operations.

Translation: For large n (at least some c_1), the function $g(n)$ dominates $f(n)$ up to a constant factor.

Examples:

- $1 \in O(n)$. This is because $1 \leq 1 \cdot n$ (so $c_2 = 1$) for all $n \geq 1 = c_1$.
- $n \in O(\frac{n}{2})$. This is because $n \leq 2 \cdot \frac{n}{2}$ (so $c_2 = 2$) for all $n \geq 1 = c_1$.

Definition 2 (Lower bound $\Omega(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in \Omega(g(n))$ if there exist constants c_1, c_2 such that for all $n \geq c_1$,

$$f(n) \geq c_2 g(n).$$

Example: $n \in \Omega(n+7)$. This is because $n \geq \frac{1}{2} \cdot (n+7)$ (so $c_2 = \frac{1}{2}$) for all $n \geq 7 = c_1$.

Definition 3 (Tight bound $\Theta(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in \Theta(g(n))$ if $f \in O(g(n))$ and $f \in \Omega(g(n))$.

Exercise: True or False?

$f(n)$	$g(n)$	$O(g(n))$	$\Omega(g(n))$	$\Theta(g(n))$
$10^6 n^3 + 2n^2 - n + 10$	n^3	T	T	T
$\sqrt{n} + \log n$	$\sqrt{n}$	T	T	T
$n(\log n + \sqrt{n})$	$\sqrt{n}$	F	T	F
n	n^2	T	F	F

Example solution: Let $f(n) = 10^6 n^3 + 2n^2 - n + 10$. For $c_2 = (10^6 + 12)$,

$$10^6 n^3 + 2n^2 - n + 10 \leq c_2 n^3$$

for all $n \geq 1$, hence it is true that $f(n) = O(n^3)$.

For $c_2 = 1$, $10^6 n^3 + 2n^2 - n + 10 \leq c_2 n^3$, hence it is true that it is $f(n) = \Omega(n^3)$.

Since $f(n) = O(n^3)$ and $f(n) = \Omega(n^3)$, then $f(n) = \Theta(n^3)$ as well.

Definition 4 (Strict upper bound $o(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in o(g(n))$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0,$$

or equivalently, for *any* constant $c_2 > 0$, there exists a constant c_1 such that for all $n \geq c_1$,

$$f(n) < c_2 g(n).$$

Definition 5 (Strict lower bound $\omega(\cdot)$)**.** For a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \in \omega(g(n))$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty,$$

or equivalently, for *any* constant $c_2 > 0$, there exists a constant c_1 such that for all $n \geq c_1$,

$$f(n) > c_2 g(n).$$

Asymptotic Properties

- Multiplication by a constant:

  If $f(n) = O(g(n))$ then for any $c > 0$, $c \cdot f(n) = O(g(n))$.

- Transitivity:

  If $f(n) = O(h(n))$ and $h(n) = O(g(n))$ then $f(n) = O(g(n))$.

- Symmetry:

  If $f(n) = O(g(n))$ then $g(n) = \Omega(f(n))$.

  If $f(n) = \Theta(g(n))$ then $g(n) = \Theta(f(n))$.

- Dominant Terms:

  If $f(n) = O(g(n))$ and $d(n) = O(e(n))$ then $f(n) + d(n) = O(\max\{g(n), e(n)\})$. It's fine to write this as $O(g(n) + e(n))$.

Common Functions

- Polynomials: $a_0 + a_1 n + \cdots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$.

- Polynomial time: Running time is $O(n^d)$ for some constant d independent of the input size n.

- Logarithms: $\log_a n = \Theta(\log_b n)$ for all constants $a, b > 0$. This means we can avoid specifying the base of the logarithm.

  For every $x > 0$, $\log n = o(n^x)$. Hence log grows slower than every polynomial.

- Exponentials: For all $r > 1$ and all $d > 0$, $n^d = o(r^n)$. Every polynomial grows slower than every exponential

- Factorial: By Sterling's formula, factorials grow faster than every exponential:

$$n! = (\sqrt{2\pi n}) \left(\frac{n}{e}\right)^n (1 + o(1)) = \Theta(n^n) = 2^{\Theta(n \log n)}.$$