DS 320 Algorithms for Data Science	**Lecture #19 Worksheet**
Spring 2023	Prof. Kira Goldner

Linear Programming II: Algorithms, Problems, and Duality

What Does Linear Programming Buy Us?

a. We know efficient algorithms exist (and have a nice theory behind them).

b. We can relate problems to one another through relaxations, duality.

c. It gives us techniques for approximation.

Linear Programming Algorithms

a. Simplex

b. Ellipsoid

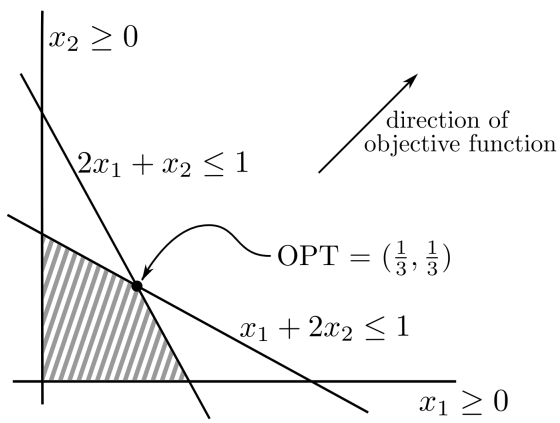


Figure 1: A toy example of a linear program.

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & 2x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \leq 1. \end{aligned}$$

Writing Problems We Know as Linear Programs

Independent Set

Given a graph $G = (V, E)$, each vertex i has weight w_i, find a maximum weighted *independent set*. S is an independent set if it does not contain both i and j for $(i, j) \in E$.

a. *Decision variables:* What are we try to solve for?

b. *Constraints:*

c. *Objective function:*

The linear program:

Integer programs vs. linear relaxations:

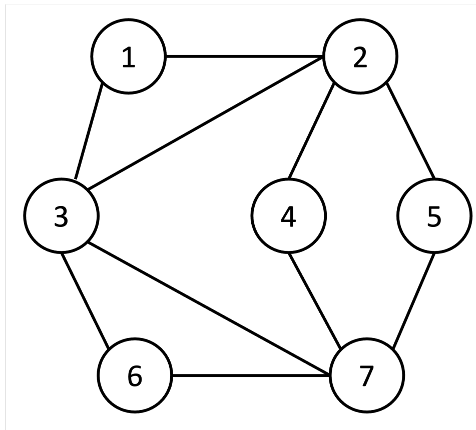
Knapsack

Given n items, each item i with value v_i and weight w_i, select a set S that contains maximum value but has total weight of at most W.

The Vertex Cover Problem

Given a graph $G = (V, E)$, we say that a set of nodes $S \subseteq V$ is a *vertex cover* if every edge $e = (i, j) \in E$ has at least one endpoint i or j in S. Our goal is to find a *minimum* vertex cover.

The decision version of the problem is: Given a graph G and a number k, does G contain a vertex cover of size at most k?



In this graph, the *minimum* vertex cover is

This is the same graph from last time when we discussed Independent Set. Do we notice any relationship? **Are there any implications of this?**

Vertex Cover as a Linear Program

a. *Decision variables:* What are we try to solve for?

b. *Constraints:*

c. *Objective function:*

Vertex Cover as a Linear Program:

Claim 1. Let S^* denote the optimal vertex cover of minimum weight, and let x^* denote the optimal solution to the Linear Program. Then $\sum_{i \in V} w_i x_i^* \leq w(S^*)$.