DS 320 Algorithms for Data Science — **Lecture #19**
Spring 2023 — Prof. Kira Goldner

Linear Programming II: Algorithms and Problems

What Does Linear Programming Buy Us?

a. We know efficient algorithms exist (and have a nice theory behind them).

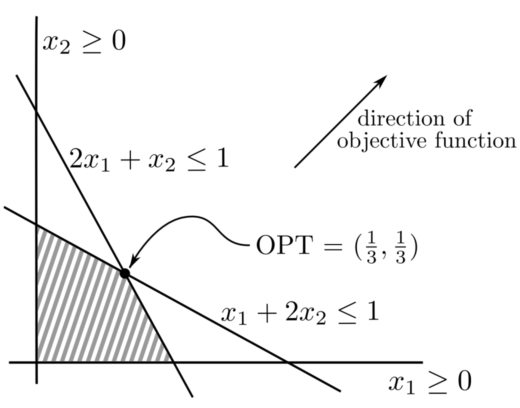
b. We can relate problems to one another through relaxations, duality.

c. It gives us techniques for approximation.

Linear Programming Algorithms

The standard algorithm for solving LPs is the Simplex Algorithm, developed in the 1940s. Its *worst-case* running time is not polynomial: you can come up with bad examples for it. But in practice, the algorithm runs pretty fast. Only much later in 1980 was it shown that linear programs could always be solved in polynomial time by the Ellipsoid Algorithm (but it tends to be slow in practice). Later on, a faster polynomial-time algorithm called Karmarkar's Algorithm was developed, which is competitive with Simplex. In practice, what you should actually do is use a commercial LP package, for instance LINDO, CPLEX, Gurobi, and Solver (in Excel). We'll just give some intuition and the high-level idea of how they work by viewing linear programming as a geometrical problem. Then we'll talk about an elegant algorithm for low-dimensional problems.

Geometry: Think of an n-dimensional space with one coordinate per variable. A solution is a point in this space. An inequality, like $x_1 + x_2 \leq 6$ is saying that we need the solution to be on a specified side of a certain hyperplane. The feasible region is the convex region in space defined by these constraints. Then we want to find the feasible point that is farthest in the "objective" direction.



$$\begin{array}{rl} \max & x_1 + x_2 \\ \text{s.t.} & x_1 \geq 0 \\ & x_2 \geq 0 \\ & 2x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \leq 1. \end{array}$$

The Simplex Algorithm: The idea is to start at some "corner" of the feasible region. Then we repeatedly do the following step: look at all neighboring corners of our current position and go to the best one (the one for which the objective function is greatest) if it is better than our current position. Stop when we get to a corner where no neighbor has a higher objective value than we currently have. The key facts here are that

1. since the objective is linear, the optimal solution will be at a corner (or maybe multiple corners), and

2. there are no local maxima: if you're not optimal, then some neighbor of you must have a strictly larger objective value than you have. That's because the feasible region is convex.

So, the Simplex method is guaranteed to halt at the best solution. The problem is that it is possible for there to be an exponential number of corners, and it is possible for Simplex to take an exponential number of steps to find the optimal corner. But, in practice this usually works well.

The Ellipsoid Algorithm: The Ellipsoid Algorithm was invented by Khachiyan in 1980 in Russia. This algorithm solves just the "feasibility problem," but you can then do binary search with the objective function to solve the optimization problem. The idea is to start with a big ellipse (called an ellipsoid in higher dimensions) that we can be sure contains the feasible region. Then, try the center of the ellipse to see if it violates any constraints. If not, you're done. If it does, then look at some constraint violated. So we know the solution (if any) is contained in the remaining at-most-half-ellipse. Now, find a new smaller ellipse that contains that half of our initial ellipse. We then repeat with the new smaller ellipse. One can show that in each step, you can always create a new smaller ellipse whose volume is smaller, by at least a $(1-1/n)$-factor, than the original ellipse. So, every n steps, the volume has dropped by about a factor of $1/e$. One can then show that if you ever get too small a volume, as a function of the number of bits used in the coefficients of the constraints, then that means there is no solution after all.

One nice thing about the Ellipsoid Algorithm is you just need to tell if the current solution violates any constraints or not, and if so, to produce one. You don't need to explicitly write them all down. There are some problems that you can write as a linear program with an exponential number of constraints if you had to write them down explicitly, but where there is an fast algorithm to determine if a proposed solution violates any constraints and if so to produce one. For these kinds of problems, the Ellipsoid Algorithm is a good one.

Writing Problems We Know as Linear Programs

Independent Set

Given a graph $G = (V, E)$, each vertex i has weight w_i, find a maximum weighted *independent set*. S is an independent set if it does not contain both i and j for $(i, j) \in E$.

a. *Decision variables:* What are we try to solve for? A set of vertices S that is independent set: our variables are x_i for each vertex i, where we want $x_i = 1$ if i is in our independent set.

$$x_i = \begin{cases} 1 & i \in S \\ 0 & \text{o.w.} \end{cases}$$

b. *Constraints:* We cannot put vertices both i and j into the independent set if they share and edge, so

$$x_i + x_j \leq 1 \quad \forall\, (i,j) \in E.$$

and we can take at most one of each vertex and no negative quantities, so

$$x_i \in [0,1] \quad \forall\, i \in V.$$

c. *Objective function:* We want to maximize the size of our independent set:

$$\max \sum_{i \in V} w_i x_i.$$

Note that this is a linear function.

We can put this all together, but that would give us an *integer program*, not a *linear program*. Asking that $x_i \in \{0,1\}$ for all i is not a linear constraint. Solving this would precisely solve independent set, which we know to be NP-Hard. Instead, we relax this constraint to a fractional constraint so that it's linear, and just ask instead that $x_i \in [0,1]$. This larger feasible region will still include all of the integer points, but will also include new fractional points, which can only have a better objective function. We call this the linear programming *relaxation*.

$$\begin{aligned} \max \sum_{i \in V} & w_i x_i & \\ \text{s.t.} \quad & x_i + x_j \leq 1 & (i,j) \in E \\ & x_i \in [0,1] & i \in V. \end{aligned}$$

Knapsack

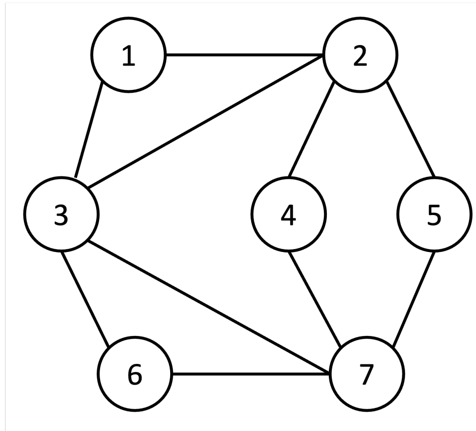
Given n items, each item i with value v_i and weight w_i, select a set S that contains maximum value but has total weight of at most W.

$$\begin{aligned} \max \sum_{i} & v_i x_i & \\ \text{s.t.} \quad & \sum_{i} w_i x_i \leq W & \\ & x_i \in [0,1] & \forall i. \end{aligned}$$

The Vertex Cover Problem

Given a graph $G = (V, E)$, we say that a set of nodes $S \subseteq V$ is a *vertex cover* if every edge $e = (i,j) \in E$ has at least one endpoint i or j in S. Our goal is to find a *minimum* vertex cover.

For the decision version of the problem, we ask: Given a graph G and a number k, does G contain a vertex cover of size at most k?



In this graph, the *minimum* vertex cover is the set of nodes {2, 3, 7} for a size of 3.

This is the same graph from last time when we discussed Independent Set. Do we notice any relationship?

Claim 1. For any graph $G = (V, E)$, S is an independent set if and only if $V \setminus S$ is a vertex cover.

Corollary 1. *Finding a maximum independent set is equivalent to finding a minimum vertex cover. Then Independent Set $\leq_P$ Vertex Cover and Vertex Cover $\leq_P$ Independent Set.*

Corollary 2. *Vertex Cover is NP-complete.*

Vertex Cover as an Integer Program

a. *Decision variables:* What are we try to solve for? A set of vertices S that is our vertex cover. So our variables are x_i for each item i, where we want $x_i = 1$ if i is in our vertex cover.

b. *Constraints:* We can never put more than 1 of a vertex into our cover, so

$$x_i \leq 1 \quad \forall i$$

and similarly, we can never take a negative quantity of an vertex, so

$$x_i + x_j \geq 1 \quad (i, j) \in E$$

Finally, we need to take at least one endpoint per edge:

$$\sum_i x_i w_i \leq W$$

c. *Objective function:* We want to minimize the size/weight of our vertex cover:

$$\max \sum_i v_i x_i$$

Note that this is again a linear function.

$$\begin{aligned}
\min \sum_{i \in V} & w_i x_i \\
\text{s.t.} \quad & x_i + x_j \geq 1 && (i,j) \in E \\
& x_i \in \{0,1\} && i \in V.
\end{aligned}$$

Vertex Cover as a Linear Program

$$\begin{aligned}
\min \sum_{i \in V} & w_i x_i \\
\text{s.t.} \quad & x_i + x_j \geq 1 && (i,j) \in E \\
& x_i \in [0,1] && i \in V.
\end{aligned}$$

Claim 2. Let S^* denote the optimal vertex cover of minimum weight, and let x^* denote the optimal solution to the Linear Program. Then $\sum_{i \in V} w_i x_i^* \leq w(S^*) = \textsc{opt}$.

Proof. The vertex cover problem is equivalent to the integer program, whereas the linear program is a *relaxation*. Then there are simply more solutions allowed to the linear program, so the minimum can only be smaller. □