DS 320 Algorithms for Data Science	**Lecture #20 Worksheet**
Spring 2023	Prof. Kira Goldner

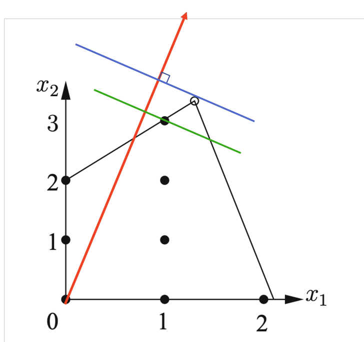
Linear Programming III: Duality

Using Linear Programming for a Vertex Cover Approximation Algorithm

$$\begin{aligned} \min \sum_{i \in V} & w_i x_i & \\ \text{s.t.} \quad & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \in [0,1] & i \in V. \end{aligned}$$

Claim 1. Let S^* denote the optimal vertex cover of minimum weight, and let x^* denote the optimal solution to the Linear Program. Then $\sum_{i \in V} w_i x_i^* \leq w(S^*)$.

Proof. The vertex cover problem is equivalent to the integer program, whereas the linear program is a *relaxation*. Then there are simply more solutions allowed to the linear program, so the minimum can only be smaller. □



$$\begin{aligned} \max \quad & 4x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 17 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Figure 1: Left: The red arrow represents the objective function, with the green line tangent to the set of feasible integer solutions, indicating the optimal integral point, and the blue line tangent to the relaxed convex feasible set, indicating the best fractional point in the relaxation, with a larger objective function. Right: The linear program for the figure on the left.

Claim 2. The set $S = \{i : x_i \geq 0.5\}$ is a vertex cover, and $w(S) \leq 2 \sum_{i \in V} w_i x_i^*$.

The Dual of a Linear Program

Every linear program has a *dual* linear program. We call the original linear program the *primal.* A maximization problem's dual is a minimization problem. There are a bunch of amazing properties that come from LP duality.

To take the dual: Label each primal constraint with a new dual variable. In our new linear program, each dual constraint will correspond to a primal variable. For the left-hand side, count up the appearances of this constraint's primal variable (e.g., x_1) in each of the primal constraints and multiply them by the dual variable for those constraints. That is, if x_1 appears 5 times ($5x_1$) in constraint for y_1, then add $5y_1$ to x_1's constraint. Don't forget to include its appearance in the primal's objective function, but this will be the right-hand side of the constraint. Finally, the dual objective function is given by the right-hand side coefficients and their correspondence to the dual variables via the constraints in the primal.

Primal:

$$\begin{array}{rll} \max & 8x_1 + 15x_2 + 3x_3 & \\ \text{subject to} & 5x_1 + 4x_2 + 2x_3 \leq 0.6 & (y_1) \\ & 7x_1 + 2x_2 + 1x_3 \leq 0.35 & (y_2) \\ & x_1, x_2, x_3 \geq 0 & \text{(non-negativity)} \end{array}$$

Dual:

The following is the normal form for a maximization problem primal and its primal:

$$\begin{array}{rl} \max & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array} \qquad \begin{array}{rl} \min & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \end{array}$$

For the above example:

$\mathbf{A} =$ $\quad$ $\mathbf{b} =$ $\quad$ $\mathbf{c} =$

Example 3: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges S such that no vertex is covered by more than one edge.

Decision variables:

Linear Program:

Taking the dual of the above primal, we get what linear program?

What problem is this?

Conditions for Optimality

Weak Duality

Theorem 1. *If* $\mathbf{x}$ *is feasible in (P) and* $\mathbf{y}$ *is feasible in (D) then* $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Proof.

Give an upper bound on maximum matching:

Give a lower bound on vertex cover: