Abstract Data Types and Depth-First Search

Let's review the main abstract data types that we might use when implementing various algorithms.

Linked Lists

Consider a list $L = [x_1, x_2, ..., x_n]$ where each x_i is an element in the list. In a *singly-linked list*, we keep a pointer to the first element of the list—that is, head $(L) = x_1$, and each element x_i has a pointer to the element after it, so next $(x_i) = x_{i+1}$ and next $(x_n) = null$.

There is no reason for singly-linked lists to be used in practice. You will never see them, with the exception of perhaps a coding interview question or a puzzle.

A doubly-linked list also has a pointer to the last element of the list $(tail(L) = x_n)$ as well as pointers form each element to the previous element $(prev(x_i) = x_{i-1} \text{ and } prev(x_1) = null)$.

It's constant to do the actual insertion or deletion of an element, and at most linear (O(n)) to find an element by starting at the head or tail and moving along the list until it is found.



Queues

Queues are what's known as First-In, First-Out (FIFO) linked lists. They support the following additional operations:

- enqueue(q, x): insert element x to the back of the queue q. Formally, $q = q \circ x$.
- dequeue(q): delete the element at the front of the queue q and return it. Formally, $q = [x_2, \ldots, x_n]$, return x_1 .

Stacks

Stacks are what's known as Last-In, First-Out (LIFO) linked lists. They support the following additional operations:

• push(s, x): insert element x to the top (back) of the stack s. Formally, $s = s \circ x$.



• pop(s): delete the element at the top (back) of the stack s and return it. Formally, $s = [x_1, \ldots, x_{n-1}]$, return x_n .



Graphs

Definition 1. A (directed) graph G = (V, E) is defined by a set of vertices V and a set of (ordered) edges $E \subseteq V \times V$.

Definition 2. A *directed edge* is an ordered pair of vertices (u, v) and is usually indicated by drawing a line between u and v, with an arrow pointing towards v.

Definition 3. An *undirected edge* is an unordered pair of vertices $\{u, v\}$ and is usually indicated by drawing a line between u and v. It indicates the existence of ordered edges (u, v) and (v, u).

Typically undirected edges will also be notated (u, v) out of sloppiness.

Some conventions:

- We will refer to the number of vertices (or the size of the vertex set |V|) as n.
- We will refer to the number of edges (or the size of the edge set |E|) as m.
- Often we will simply name the vertices $V = \{1, ..., n\}$ so an edge (i, j) is an edge from the i^{th} vertex to the j^{th} vertex.



Figure 1: Left: An example undirected graph. $V = \{1, 2, 3\}$. $E = \{(1, 2)\}$. Right: An example directed graph. $V = \{1, 2, 3, 4, 5, 6, 7\}$. $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7)\}$.

• You may also hear vertices referred to as "nodes" or edges referred to as "arcs."

Definition 4. We call vertices *i* and *j* adjacent or neighbors if there is an edge $(i, j) \in E$. In directed graphs, we may explicitly refer to out-neighbors ($\{j : (i, j) \in E\}$) or in-neighbors ($\{j : (j, i) \in E\}$).

Definition 5. The degree of a vertex v is the number of neighbors it has. That is, $d_v = |\{u : (v, u) \in E\}|$. For directed graphs, we may refer to a vertex's *in-degree* or *out-degree*, and its *degree* is the sum of these.

Definition 6. A path from u to w is a sequence of edges e_1, e_2, \ldots, e_k such that $e_1 = (u, v_1), e_i = (v_{i-1}, v_i)$, and $e_k = (v_{k-1}, w)$. That is, the first edge starts at u, the last edge ends at w, and each proceeding edge ends where the previous edge starts.

Definition 7. We say that a pair of vertices are *connected* if there exists a path between them.

We see graphs all over; networks are an entire field of study! What can you represent with graphs?

- Transportation networks (roads, airlines)
- Communication networks (Bitcoin peer-to-peer network)
- Information network (internet with links)
- Social networks
- Dependency network (course prerequisites, food chain)

What graph problems do you know?

- Shortest path
- Traveling salesman
- Scheduling

Abstract Data Types for Graphs

There are two primary ways that we represent graphs in the computer.

Exercise: Ask yourself the following questions for both adjacency matrices and adjacency lists to fill out the pros and cons (below) for each graph ADT below:

- What is the worst-case runtime to look up a specific edge (i, j)?
- What is the worst-case space needed to store the graph?
- What is the runtime to list all edges adjacent to i? On average, per edge adjacent to i?

Definition 8. An *adjacency matrix* for G = (V, E) is an $n \times n$ binary matrix A where $A_{ij} = 1$ if and only if $(i, j) \in E$. We use a 2-dimensional array.

Pros of using an adjacency matrix:

• Look-up of a specific (i, j) edge is O(1).

Cons of using an adjacency matrix:

- Space is $\Omega(n^2)$, independent of m. This can be very wasteful for sparse graphs where m is small.
- Listing all of i's edges is $\Omega(n)$ time, which can again be wasteful if i has small degree.

Definition 9. An *adjacency list* for G = (V, E) is an array A of length n where the i^{th} entry contains a linked list of i's neighbors. That is, j is in the list A[i] if and only if $(i, j) \in E$.

Pros of using an adjacency list:

- Listing all of *i*'s edges is $O(d_i)$ time, hence O(1) per neighbor.
- Space is O(n+m).

Cons of using an adjacency list:

• Look-up of a specific (i, j) edge is $O(d_i) = O(n)$.