DS 320 Algorithms for Data Science	Lecture #20 Worksheet
Spring 2025	Prof. Kira Goldner and Jeffrey Considine

Linear Programming III: Duality Theory and Zero-Sum Games

Conditions for Optimality

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \end{aligned}$$

Weak Duality

Theorem 1. *If $\mathbf{x}$ is feasible in (P) and $\mathbf{y}$ is feasible in (D) then $\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}$.*

Give an upper bound on maximum matching:

Give a lower bound on vertex cover:

Strong Duality

Theorem 2 (Strong Duality)**.** *A pair of solutions $(\mathbf{x}^*, \mathbf{y}^*)$ are optimal for the primal and dual respectively if and only if $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.*

Proof. ($\Rightarrow$) Skip.
($\Leftarrow$)

Complementary Slackness

Primal (P):

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \sum_i a_{ji} x_i \le b_j \quad \forall j \quad (y_j) \\ & x_i \ge 0 \quad \forall i \end{aligned}$$

Dual (D):

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \sum_i a_{ij} y_i \ge c_i \quad \forall i \quad (x_i) \\ & y_j \ge 0 \quad \forall j \end{aligned}$$

Theorem 3 (Complementary Slackness). *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if the following complementary slackness conditions (1) and (2) hold:*

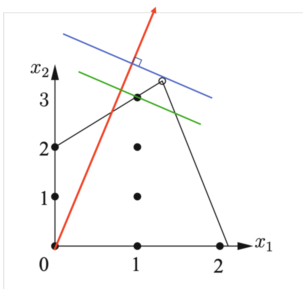
Proof.

Using Linear Programming for a Vertex Cover Approximation

$$\begin{aligned} &\min \sum_{i \in V} w_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 && (i,j) \in E \\ & x_i \in [0,1] && i \in V. \end{aligned}$$

Claim 1. Let S^* denote the optimal vertex cover of minimum weight, and let x^* denote the optimal solution to the Linear Program. Then $\sum_{i \in V} w_i x_i^* \leq w(S^*)$.

Proof. The vertex cover problem is equivalent to the integer program, whereas the linear program is a *relaxation.* Then there are simply more solutions allowed to the linear program, so the minimum can only be smaller. □



$$\begin{aligned} \max \quad & 4x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 17 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Claim 2. The set $S = \{i : x_i \geq 0.5\}$ is a vertex cover, and $w(S) \leq 2 \sum_{i \in V} w_i x_i^*$.

Zero-Sum Games and the Minimax Theorem

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

The Minimax Theorem

Theorem 4 (Minimax Theorem)**.** *For every two-player zero-sum game* $\mathbf{A}$*,*

$$\max_{\mathbf{x}} \left(\min_{\mathbf{y}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right) = \min_{\mathbf{y}} \left(\max_{\mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right). \tag{1}$$