DS 320 Algorithms for Data Science	**Lecture #3 Worksheet**
Spring 2025	Prof. Kira Goldner and Jeffrey Considine

Loop Invariants and Induction

Definition 1. A *loop invariant* is something that is true before we start and after every iteration of a loop.

We prove that a loop invariant is true by showing the following three things about it:

- **Initialization:** It is true prior to the first iteration of the loop.
- **Maintenance:** If it is true before an iteration of the loop, it remains true before the next iteration.
- **Termination:** When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

Algorithm 1 add(A).

Input: A is an array of integers. It is indexed 1 to n.
sum $= 0$
for $i = 1$ to n **do**
  sum $+ = A[i]$
end for
return sum

Claim 1.

Proof. We will prove this formally as a loop invariant.

Initialization: Before the first iteration of the "for" loop,

Maintenance: If our statement holds before an iteration of the loop, then

Termination: When the loop terminates,

□

Induction

Now we'll show how to prove the same thing via induction.

Proof. We show the following by induction on i: (*premise*)

Base Case:

Inductive Hypothesis:

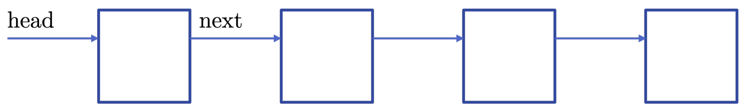
Inductive Step:

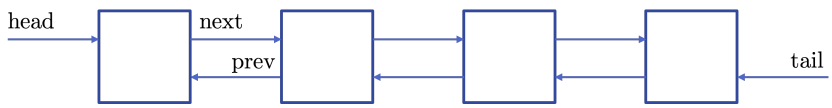
□

Abstract Data Types and Depth-First Search

Linked Lists

Consider a list $L = [x_1, x_2, \ldots, x_n]$ where each x_i is an element in the list. We keep a pointer to the head (and the tail) of the list. Each element x_i has a pointer "next" (and "previous").



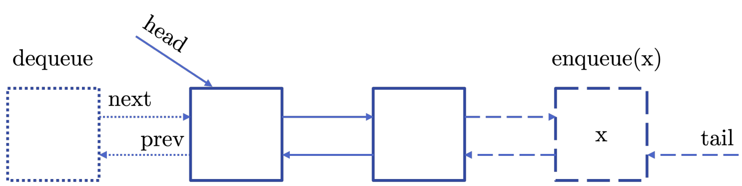


- What is the (worst-case) runtime to find an element?
- What is the (worst-case) runtime to insert or delete an element (once it's found)?

Queues

Queues are First-In, First-Out (FIFO) linked lists. They support the operations:

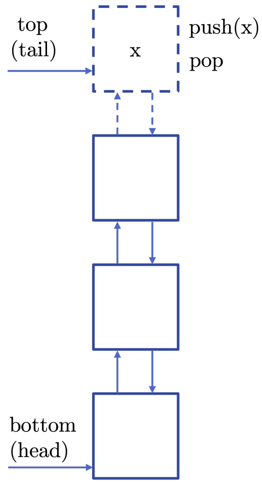
- enqueue(q, x): insert element x to the back of the queue q. Formally, $q = q \circ x$.
- dequeue(q): delete the element at the front of the queue q and return it. Formally, $q = [x_2, \ldots, x_n]$, return x_1.



Stacks

Stacks are what's known as Last-In, First-Out (LIFO) linked lists. They support the operations:

- push(s, x): insert element x to the top (back) of the stack s. Formally, $s = s \circ x$.
- pop(s): delete the element at the top (back) of the stack s and return it. Formally, $s = [x_1, \ldots, x_{n-1}]$, return x_n.



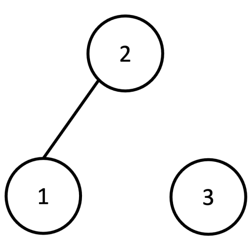
Graphs

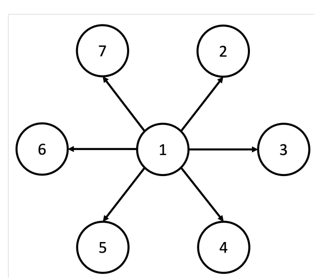
Definition 2. A (directed) *graph* $G = (V, E)$ is defined by a set of vertices V and a set of (ordered) edges $E \subseteq V \times V$.

Definition 3. A *directed edge* is an ordered pair of vertices (u, v) and is usually indicated by drawing a line between u and v, with an arrow pointing towards v.

Definition 4. An *undirected edge* is an unordered pair of vertices $\{u, v\}$ and is usually indicated by drawing a line between u and v. It indicates the existence of ordered edges (u, v) and (v, u).

Typically undirected edges will also be notated (u, v) out of sloppiness.



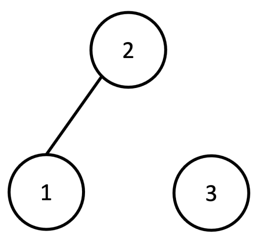


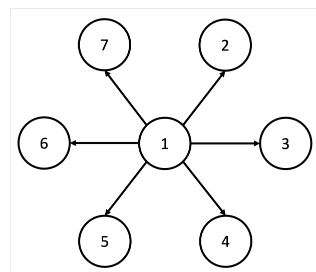
Some conventions:

- We will refer to the number of vertices (or the *size* of the vertex set $|V|$) as n.
- We will refer to the number of edges (or the *size* of the edge set $|E|$) as m.
- Often we will simply name the vertices $V = \{1, \ldots, n\}$ so an edge (i, j) is an edge from the i^{th} vertex to the j^{th} vertex.
- You may also hear vertices referred to as "nodes" or edges referred to as "arcs."

Definition 5. We call vertices i and j *adjacent* or *neighbors* if there is an edge $(i, j) \in E$. In directed graphs, we may explicitly refer to *out-neighbors* ($\{j : (i, j) \in E\}$) or *in-neighbors* ($\{j : (j, i) \in E\}$).

Definition 6. The *degree* of a vertex v is the number of neighbors it has. That is, $d_v = |\{u : (v, u) \in E\}|$. For directed graphs, we may refer to a vertex's *in-degree* or *out-degree*, and its *degree* is the sum of these.





Definition 7. A *path* from u to w is a sequence of edges $e_1, e_2, \ldots, e_k$ such that $e_1 = (u, v_1), e_i = (v_{i-1}, v_i)$, and $e_k = (v_{k-1}, w)$. That is, the first edge starts at u, the last edge ends at w, and each proceeding edge ends where the previous edge starts.

Definition 8. We say that a pair of vertices are *connected* if there exists a path between them.

We see graphs all over; networks are an entire field of study! What can you represent with graphs?

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What graph problems do you know?

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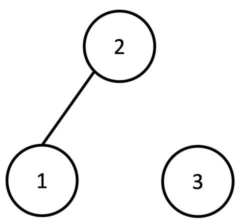
Abstract Data Types for Graphs

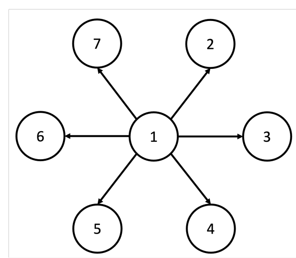
There are two primary ways that we represent graphs in the computer.

Definition 9. An *adjacency matrix* for $G = (V, E)$ is an $n \times n$ binary matrix A where $A_{ij} = 1$ if and only if $(i, j) \in E$.

Pros of using an adjacency matrix:

Cons of using an adjacency matrix:





Definition 10. An *adjacency list* for $G = (V, E)$ is an array A of length n where the i^{th} entry contains a linked list of i's neighbors. That is, j is in the list $A[i]$ if and only if $(i, j) \in E$.

Pros of using an adjacency list:

Cons of using an adjacency list:

Exercise: Ask yourself the following questions for both adjacency matrices and adjacency lists to fill out the pros and cons (above) for each graph ADT above:

- What is the worst-case runtime to look up a specific edge (i, j)?
- What is the worst-case space needed to store the graph?
- What is the runtime to list all edges adjacent to i? On average, per edge adjacent to i?