Linear Programming III: Duality Theory and Zero-Sum Games

Linear Programming Duality

The Dual of a Linear Program

Every linear program has a *dual* linear program. We call the original linear program the *primal*. A maximization problem's dual is a minimization problem. There are a bunch of amazing properties that come from LP duality.

To take the dual: Label each primal constraint with a new dual variable. In our new linear program, each dual constraint will correspond to a primal variable. For the left-hand side, count up the appearances of this constraint's primal variable (e.g., x_1) in each of the primal constraints and multiply them by the dual variable for those constraints. That is, if x_1 appears 5 times ($5x_1$) in constraint for y_1, then add $5y_1$ to x_1's constraint. Don't forget to include its appearance in the primal's objective function, but this will be the right-hand side of the constraint. Finally, the dual objective function is given by the right-hand side coefficients and their correspondence to the dual variables via the constraints in the primal.

Primal:

$$\begin{aligned} \max \quad & 8x_1 + 15x_2 + 3x_3 & \\ \text{subject to} \quad & 5x_1 + 4x_2 + 2x_3 \leq 0.6 & (y_1) \\ & 7x_1 + 2x_2 + 1x_3 \leq 0.35 & (y_2) \\ & x_1, x_2, x_3 \geq 0 & \text{(non-negativity)} \end{aligned}$$

Dual:

The following is the normal form for a maximization problem primal and its dual:

$$\begin{array}{rl} \max & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{array} \qquad\qquad \begin{array}{rl} \min & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \end{array}$$

For the above example:

$\mathbf{A} =$ $\qquad$ $\mathbf{b} =$ $\qquad$ $\mathbf{c} =$

Example: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges S such that no vertex is covered by more than one edge.

Decision variables:

Linear Program:

Taking the dual of the above primal, we get what linear program?

What problem is this?

Weak Duality

Theorem 1. *If* $\mathbf{x}$ *is feasible in (P) and* $\mathbf{y}$ *is feasible in (D) then* $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Give an upper bound on maximum matching:

Give a lower bound on vertex cover:

Strong Duality

Theorem 2 (Strong Duality)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if* $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.

Proof. $(\Rightarrow)$ Skip.
$(\Leftarrow)$

Complementary Slackness

Primal (P):

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \sum_i a_{ji} x_i \leq b_j \quad \forall j \quad (y_j) \\ & x_i \geq 0 \quad \forall i \end{aligned}$$

Dual (D):

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \sum_i a_{ij} y_i \geq c_i \quad \forall i \quad (x_i) \\ & y_j \geq 0 \quad \forall j \end{aligned}$$

Theorem 3 (Complementary Slackness)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if the following complementary slackness conditions (1) and (2) hold:*

Proof.

Zero-Sum Games and the Minimax Theorem

Notation:

- $m \times n$ payoff matrix $\mathbf{A}$—a_{ij} is the row player's payoff for outcome (i, j) when row player plays strategy i and column player plays strategy j
- mixed row strategy $\mathbf{x}$ (a distribution over rows)
- mixed column strategy $\mathbf{y}$ (a distribution over columns)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Expected payoff of the row player:

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n} \Pr[\text{outcome } (i,j)]\, a_{ij} &= \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{\Pr[\text{row } i \text{ chosen}]}_{=x_i}\, \underbrace{\Pr[\text{column } j \text{ chosen}]}_{=y_j}\, a_{ij} \\ &= \mathbf{x}^T \mathbf{A} \mathbf{y}\end{aligned}$$

Theorem 4 (Minimax Theorem)**.** *For every two-player zero-sum game* $\mathbf{A}$*,*

$$\max_{\mathbf{x}} \left(\min_{\mathbf{y}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right) = \min_{\mathbf{y}} \left(\max_{\mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right). \tag{1}$$

Or in English, the expected payoff of the row play is the same whether the row player goes first or second. This is called the *value of the game.*

From LP Duality to Minimax

Issues:

(1)

(2)

Observation:

$$\max_{\mathbf{x}} \left(\min_{\mathbf{y}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right) = \max_{\mathbf{x}} \left(\min_{j=1}^{n} \mathbf{x}^T \mathbf{A} \mathbf{e}_j \right) \tag{2}$$

$$= \max_{\mathbf{x}} \left(\min_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i \right) \tag{3}$$

where $\mathbf{e}_j$ is the j^{th} standard basis vector:

$$(\mathbf{e}_j)_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

$$\max v$$

subject to

$$v - \sum_{i=1}^{m} a_{ij} x_i \leq 0 \quad \text{for all } j = 1, \ldots, n$$

$$\sum_{i=1}^{m} x_i = 1$$

$$x_1, \ldots, x_m \geq 0 \quad \text{and} \quad v \in \mathbb{R}.$$

$$\min w$$

subject to

$$w - \sum_{j=1}^{n} a_{ij} y_j \geq 0 \quad \text{for all } i = 1, \ldots, m$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_1, \ldots, y_n \geq 0 \quad \text{and} \quad w \in \mathbb{R}.$$