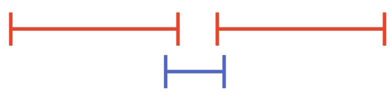
DS 320 Algorithms for Data Science	**Lecture #7**
Spring 2026	Prof. Kira Goldner and Jeffrey Considine

Greedy II: Interval Scheduling

Suppose you are given n jobs to schedule on a machine. Each job i (where $i \in \{1, \ldots, n\}$) has a start time $s(i)$ and a finish time $f(i)$. You would like to schedule *as many* jobs as possible given that the machine can only process one job at a time, and the jobs must run from their start time to finish time uninterrupted to be processed. That is, the machine cannot process two jobs that overlap.

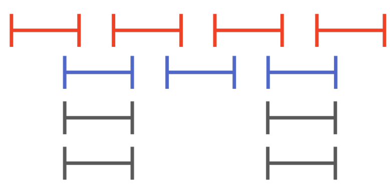
What *greedy* algorithm should you use to schedule the jobs? By what metric is it greedy? (See **Step 2**.) Here are some ideas.

- Shortest jobs. Counterexample, with red optimal solution and blue greedy solution:



- Earliest start time. Counterexample, with red optimal solution and blue greedy solution:

- Fewest conflicts. Counterexample, with red optimal solution and blue greedy solution:



- Earliest finish time. This is the correct metric that we will prove is optimal.

Prove that your algorithm is optimal by a **Greedy-Stays-Ahead** proof.

Step 1: Define your solutions. Describe the form your greedy solution takes, and what form some other solution takes (possibly the optimal solution). For example, let A be the solution constructed by the greedy algorithm, and let O be a (possibly optimal) solution.

Let $A = \{i_1, \ldots, i_k\}$ be the set of requests selected by our greedy algorithm, in the order in which they were added. Let $O = \{j_1, \ldots, j_m\}$ be the requests selected by an optimal solution, ordered by their finish times.

Step 2: Find a measure. Find a *measure* by which greedy stays ahead of the other solution you chose to compare with. Let $a_1, \ldots, a_k$ be the first k measures of the greedy algorithm, and let $o_1, \ldots, o_m$ be the first m measures of the other solution ($m = k$ sometimes).

We will compare A and O by their jobs' finish times, that is, we define the measures $a_r = f(i_r)$ and $o_r = f(j_r)$ for all $r \leq k$, and we show that for all $r \leq k$, $a_r \leq o_r$ (i.e. that $f(i_r) \leq f(j_r)$). This can be shown by induction on r.

Step 3: Prove greedy stays ahead. Show that the partial solutions constructed by greedy are always just as good as the initial segments of your other solution, based on the measure you selected.

- For all indices $r \leq \min(k, m)$, prove (often by induction) that $a_r \geq o_r$ or that $a_r \leq o_r$, whichever the case may be. Don't forget to use your algorithm to help you argue the inductive step.

Formally, for all $r \leq k$, we will prove the claim that $a_r \leq o_r$ by induction. (If you want, you can call this $P(r)$, the property for r that we aim to prove.) We want to show that this ($P(r)$) is true for all $1 \leq r \leq k$.

Base Case ($r = 1$): Since the algorithm selects the job with the earliest finish time, it must be the case that $a_1 = f(i_1) \leq f(j_1) = o_1$.

Inductive Hypothesis: Suppose that the claim holds for some fixed $r - 1$ with $r > 1$, that is, that $a_{r-1} = f(i_{r-1}) \leq f(j_{r-1}) = o_{r-1}$.

Inductive Step ($r =$: Now we prove that $P(r)$ is true using the IH that $P(r-1)$ is true. That is, we prove that $a_r \leq o_r$. Recall that by the inductive hypothesis, $f(i_{r-1}) \leq f(j_{r-1})$, and so any jobs that are compatible with the first $r - 1$ jobs in the optimal solutions are certainly compatible with the first $r - 1$ jobs of our greedy solution. Therefore, we could add j_r to our greedy solution, and since we take the compatible job with the smallest finish time, it must be the case that $f(i_r) \leq f(j_r)$, that is, that $a_r \leq o_r$, as desired.

Thus we have shown that for all $r \leq k$, $f(i_r) \leq f(j_r)$.

Step 4: Prove optimality. Prove that since greedy stays ahead of the other solution with respect to the measure you selected, then it is optimal.

Our inductive claim implies that, in particular, $f(i_k) \leq f(j_k)$. If A is not optimal, then it must be the case that $m > k$, and so there is a job j_{k+1} in O that is not in A. This job must start after O's k^{th} job finishes at time $f(j_k)$, and hence after $f(i_k)$. But then this job is compatible with all the jobs in A, and so A would have added it during the greedy algorithm. This is a contradiction, and thus A has as many elements as O.

Step 5: Analyze runtime. This is always our last step.

The algorithm begins by sorting the n requests in order of finishing time, which takes time $O(n \log n)$. Each time we select an interval, we proceed past any incompatible intervals in our list; that is, we proceed through our list exactly once. This part of the algorithm takes time $O(n)$; therefore, the total running time is $O(n \log n)$.