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| **DS 574 Algorithmic Mechanism Design** | **Lecture #21** |
| **Fall 2022** | Prof. Kira Goldner |

# Interdependent Values I

Thus far, we have been discussing private independent values. That is, each bidder $i$ has private information $\mathbf{v}_i$ regarding their value for item $i$.

However, in many settings, there valuations may be correlated between buyers, depend on one another's information, or even be common.

**The Interdependent Values Model [2].** Each bidder has a private *signal* $s_i$ that is a piece of information about the item, so in total the information about the item is $s_1, \ldots, s_n$, but is distributed amongst the different buyers. Each buyer has a **public** *valuation function* $v_i(s_1, \cdots, s_n)$ that dictates how the buyer aggregates the information into a value for the item.

Example: Common values [4]: The average of estimates $v_i(s_1, \ldots, s_n) = \frac{1}{n}\sum_i s_i \, \forall i$, or the wallet game $v_i(s_1, \ldots, s_n) = \sum_i s_i \, \forall i$.

## Optimal Social Welfare

**Mechanisms.** How can we maximize social welfare in this setting, optimally? What does a mechanism even look like?

- Report: A bid of a signal $b_i$ for each bidder $i$, truthful when $b_i = s_i$.
- Calculate: $v_i(\mathbf{b})$ for each bidder $i$
- Allocate to: [This is the decision of the mechanism.]

**Incentive Compatibility.** What conditions are necessary for maximizing social welfare optimally to be incentive-compatible? What definition of incentive-compatible are we going for?

Give an example showing why we can't expect our mechanisms to be DSIC.

So the next best we can hope for is EPIC. In this context that means:

**Definition 1.** Truth-telling is said to be ex-post Nash if, for every bidder $i$, for every possible realization of the other bidders' signals $\mathbf{s}_{-i}$, and given that other bidders report their signals truthfully, then it is in bidder $i$' best interest to report her true signal.

What is the analogue of Myerson's Lemma in the interdependent setting?

**Lemma 1** (Myerson Analogue [3]). *For every interdependent values setting,*

*(a) An allocation rule* $\mathbf{x}$ *is implementable as EPIC and ex post IR if and only if for every* $i, \mathbf{s}_{-i}$*, the allocation rule* $x_i$ *is monotone non-decreasing in the signal* $s_i$*.*

*(b) If* $\mathbf{x}$ *is monotone, then there is a unique payment rule such that the sealed-bid mechanism* $(\mathbf{x}, \mathbf{p})$ *is EPIC and ex-post IR.*

*(c) The payment rule in is given by:*

$$\begin{aligned} p_i(\mathbf{s}) = x_i(\mathbf{s})v_i(\mathbf{s}) - &\int_{v_i(0,\mathbf{s}_{-i})}^{v_i(s_i,\mathbf{s}_{-i})} x_i(v_i^{-1}(t \mid \mathbf{s}_{-i}), \mathbf{s}_{-i})dt \\ &- \left[x_i(0,\mathbf{s}_{-i})v_i(0,\mathbf{s}_{-i}) - p_i(0,\mathbf{s}_{-i})\right]; \\ p_i(0,\mathbf{s}_{-i}) \leq x_i(0,\mathbf{s}_{-i})v_i(0,\mathbf{s}_{-i}).& \end{aligned}$$

**Derivation.** Fix a bidder $i$ with public valuation function $v_i(\cdot)$. Let $s_{i\ell}$ be the $\ell^{th}$ possible realization of $s_i$ in the discrete support of $i$'s signals. Fix the signals of the other bidders $\mathbf{s}_{-i}$, and we discuss the possible values of bidder $i$ in the context of the support of the values $\{s_{i0} = 0, s_{i1}, \ldots, s_{ik}\}$ for some high $k$.

For notational brevity, in the following derivation, we drop the $\mathbf{s}_{-i}$ in the input, writing just $v_i(s_i)$, $x_i(s_i)$, and $p_i(s_i)$ instead of $v_i(s_i, \mathbf{s}_{-i})$, $x_i(s_i, \mathbf{s}_{-i})$, and $p_i(s_i, \mathbf{s}_{-i})$. Then using the fact that we seek an EPIC mechanism, we deduce the following.

The bidder with signal $s_{i\ell}$ prefers truthful reporting to reporting $s_{i\ell-1}$:

$$v_i(s_{i\ell})x_i(s_{i\ell}) - p_i(s_{i\ell}) \geq v_i(s_{i\ell})x_i(s_{i\ell-1}) - p_i(s_{i\ell-1})$$

The bidder with signal $s_{i\ell-1}$ prefers truthful reporting to reporting $s_{i\ell}$:

$$v_i(s_{i\ell-1})x_i(s_{i\ell-1}) - p_i(s_{i\ell-1}) \geq v_i(s_{i\ell-1})x_i(s_{i\ell}) - p_i(s_{i\ell})$$

Thus, this gives that:

$$\begin{aligned} v_i(s_{i\ell})\left[x_i(s_{i\ell}) - x_i(s_{i\ell-1})\right] &\geq p_i(s_{i\ell}) - p_i(s_{i\ell-1}) \\ &\geq v_i(s_{i\ell-1})\left[x_i(s_{i\ell}) - x_i(s_{i\ell-1})\right]. \end{aligned}$$

Under the assumption that $s_{i0} = 0$, this gives

$$\frac{\partial}{\partial s_i} p_i(s_i, \mathbf{s}_{-i}) \geq v_i(s_i, \mathbf{s}_{-i}) \frac{\partial}{\partial s_i} x_i(s_i, \mathbf{s}_{-i})$$

and hence

$$\begin{aligned}
p_i(s_i, \mathbf{s}_{-i}) &= \int_0^{s_i} \frac{\partial}{\partial z} p_i(z, \mathbf{s}_{-i})\, dz + p_i(0, \mathbf{s}_{-i}) \\
&= \int_0^{s_i} v_i(z, \mathbf{s}_{-i}) \frac{\partial}{\partial s_i} x_i(z, \mathbf{s}_{-i})\, dz + p_i(0, \mathbf{s}_{-i}) \\
&= x_i(\mathbf{s}) v_i(\mathbf{s}) - x_i(0, \mathbf{s}_{-i}) v_i(0, \mathbf{s}_{-i}) - \int_0^{s_i} x_i(z, \mathbf{s}_{-i}) \frac{\partial}{\partial z} v_i(z, \mathbf{s}_{-i})\, dz + p_i(0, \mathbf{s}_{-i}) \\
&= x_i(\mathbf{s}) v_i(\mathbf{s}) - \int_0^{s_i} x_i(z, \mathbf{s}_{-i}) \frac{\partial}{\partial z} v_i(z, \mathbf{s}_{-i})\, dz - [x_i(0, \mathbf{s}_{-i}) v_i(0, \mathbf{s}_{-i}) - p_i(0, \mathbf{s}_{-i})] \\
&= x_i(\mathbf{s}) v_i(\mathbf{s}) - \int_{v_i(0, \mathbf{s}_{-i})}^{v_i(s_i, \mathbf{s}_{-i})} x_i(v_i^{-1}(t \mid \mathbf{s}_{-i}), \mathbf{s}_{-i}) dt - [x_i(0, \mathbf{s}_{-i}) v_i(0, \mathbf{s}_{-i}) - p_i(0, \mathbf{s}_{-i})]\,.
\end{aligned}$$

And we need to also ensure ex-post individual rationality for the type with signal 0:

$$p_i(0, \mathbf{s}_{-i}) \leq x_i(0, \mathbf{s}_{-i}) v_i(0, \mathbf{s}_{-i}).$$

This is typically guaranteed by setting $p(0) = 0$ in the independent private value setting, but $s_i = 0$ doesn't mean that $v_i(0, \mathbf{s}_{-i}) = 0$. Guaranteeing it for the type with signal 0 ensures it for the rest of the types by the payment identity (which ensures EPIC among types).

What allocation will maximize social welfare?

The Vickrey Auction modified for the interdependent setting: collect signals, compute values, and allocate to the buyer with the highest value.

**Payments.** What are the payments?

Fix the signals of other bidders $\mathbf{s}_{-i}$. When bidder $i$ is the winner, they pay their value at their *critical signal* $s_i^*$. That is, at the signal $s_i^*(\mathbf{s}_{-i})$ where they begin winning (being the highest bidder), $s_i^* = \min\{z \mid x_i(v_i(z, \mathbf{s}_{-i})) = 1\}$.

**Truthfulness.** Is this mechanism EPIC? When might it not be?

When $x_i(\cdot)$ is not monotone in $s_i$ for some $i$—when, for some $\mathbf{s}_{-i}$, as $i$ increases their signal, they go from being the highest bidder to not the highest bidder.

**Assumptions.** What assumption could we place on the class of valuations to ensure that the mechanism is always EPIC?

Once $i$ is the highest-valued bidder, then as they increase their signal $s_i$, they remain the highest bidder. We call this *single-crossing*—once they *cross* the other bidders' values and become the highest, they never cross back down to become lower than the highest. This is *precisely* the condition that makes Vickrey truthful.

More specifically, we will ask that bidder $i$ is most sensitive to their own signal. Formally, the *single-crossing condition* requires that for all bidders $i$ and $j$,

$$\frac{\partial}{\partial s_i} v_i(s_i, \mathbf{s}_{-i}) \geq \frac{\partial}{\partial s_i} v_j(s_i, \mathbf{s}_{-i}).$$

## Beyond Single-Crossing [1]

What happens when we don't have single-crossing? Can we at least guarantee some approximation to social welfare?

**Example.** [Impossibility for deterministic prior-free mechanisms without SC.] Consider a scenario with two bidders (bidder 1 and bidder 2), where $S_1 = \{0, 1\}$ and $S_2 = \{0\}$, and the following valuation functions:

$$\begin{aligned} v_1(s_1 = 0, s_2 = 0) &= r; \qquad & v_1(s_1 = 1, s_2 = 0) &= r; \\ v_2(s_1 = 0, s_2 = 0) &= 1; \qquad & v_2(s_1 = 1, s_2 = 0) &= r^2. \end{aligned}$$

It is easy to see that $v_1$ does not satisfy single-crossing since when $s_1$ increases, $v_1$ does not increase but $v_2$ increases by $r^2 - 1$, making $v_1$ go from being $r$ times greater than $v_2$ to being $r$ times smaller than it.

We claim that, for these valuations, no truthful, deterministic, and prior-free mechanism has an approximation ratio better than $r$. To see this, consider the signal profile $(s_1 = 0, s_2 = 0)$. To get a better than $r$-approximation for this profile, bidder 1 must win the item. Truthfulness requires the allocation to be monotone in each bidder's signal, hence bidder 1 must also win at report $(s_1 = 1, s_2 = 0)$, which results in an allocation that is a factor of $r$ off from the optimal allocation. Since $r$ is arbitrary, the approximation ratio is arbitrarily bad.

**Example.** [Impossibility result for randomized mechanisms without SC.] Consider the case where every bidder has the following signal distribution for some small $\varepsilon > 0$,

$$s_i = \begin{cases} 1 & w.p.\ \varepsilon \\ 0 & w.p.\ 1 - \varepsilon, \end{cases}$$

and each agent $i$ has a valuation $v_i(\mathbf{s}) = \prod_{j \neq i} s_j$; that is, the bidder has a value 1 if and only if every other agent has signal 1. The optimal expected welfare is 1 whenever at least $n - 1$ bidders have a 1 signal. This happens with probability $\varepsilon^n + n \cdot \varepsilon^{n-1}(1 - \varepsilon)$. Therefore,

$$\text{OPT} \quad = \quad \varepsilon^n + n \cdot \varepsilon^{n-1}(1 - \varepsilon) \quad > \quad n\varepsilon^{n-1}(1 - \varepsilon). \tag{1}$$

Consider any truthful mechanism at profile $(s_i = 0, \mathbf{s}_{-i} = \mathbf{1})$. At this profile, the mechanism gets bidder $i$'s value in welfare with probability that he is allocated, $x_i(s_i = 0, \mathbf{s}_{-i} = \mathbf{1})$, and otherwise gets zero since no other bidder has non-zero value. By monotonicity, for every $i$, we have that $x_i(s_i = 0, \mathbf{s}_{-i} = \mathbf{1}) \leq x_i(\mathbf{1})$, and by feasibility, $\sum_i x_i(\mathbf{1}) \leq 1$. Under any other profile (where at least two signals are 0), all agents have zero value, so welfare is zero. The expected welfare of any truthful mechanism is thus bounded by

$$\begin{aligned}
\textsc{Welfare} &= \sum_i \Pr[s_i = 0, \mathbf{s}_{-i} = \mathbf{1}] \cdot x_i(s_i = 0, \mathbf{s}_{-i} = \mathbf{1}) \cdot 1 + \Pr[\mathbf{s} = \mathbf{1}] \sum_i x_i(\mathbf{1}) \cdot 1 \\
&= \sum_i \varepsilon^{n-1}(1-\varepsilon) \cdot x_i(s_i = 0, \mathbf{s}_{-i} = \mathbf{1}) + \varepsilon^n \sum_i x_i(\mathbf{1}) \\
&\leq \varepsilon^{n-1}(1-\varepsilon) \sum_i x_i(\mathbf{1}) + \varepsilon^n \sum_i x_i(\mathbf{1}) \\
&\leq \varepsilon^{n-1}(1-\varepsilon) + \varepsilon^n \\
&= \varepsilon^{n-1}. \qquad (2)
\end{aligned}$$

Combining (1) with (2), we get that the approximation ratio of any monotone mechanism is $\textsc{Welfare}/\textsc{opt} \leq \frac{1}{n(1-\varepsilon)}$ which can be made arbitrarily close to $1/n$; this is the same as the welfare attained by just allocating to a random bidder.

**A Restricted Class.** Optimal welfare is not attainable for general valuations. For what *natural* restricted class of valuations can we achieve some $\alpha$-approximation to optimal social welfare for every profile of signals $\mathbf{s}$ (prior-free) with an EPIC mechanism?

# References


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[4] Robert B Wilson. Competitive bidding with asymmetric information. *Management Science*, 13(11):816–820, 1967.