DS 574 Algorithmic Mechanism Design	**Lecture #23 Worksheet**
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Selling Separately or Bundling

Consider the setting with a revenue-maximizing monopolist who has two heterogenous items to sell (an apple and an orange) and a single additive buyer with values v_1 and v_2 for item 1 and 2 respectively. What mechanism should the monopolist use?

Two of the simplest possible mechanisms are as follows:

- (SREV) Sell the items separately. Post a price of p_i on each item i, optimizing these prices to maximize expected revenue from your distribution. The revenue from optimally selling items separately SREV $\geq \sum_i p_i Pr_{v_i}[v_i \geq p_i]$ for valid prices p_i.
- (BREV) Sell the items together in one grand bundle as if they were a single item with some price p. Optimize this price to maximize expected revenue from your distribution. The revenue from optimally selling the grand bundle BREV $\geq p \cdot \Pr_v[\sum_i v_i \geq p]$.

Why Getting the Optimal Mechanism is Tricky

Example 1: Bundling Can Be Better. Suppose $v_1, v_2 \sim U\{1, 2\}$.
Selling separately revenue:

Grand bundle revenue:

Example 2: Better than Bundling. Suppose $v_1, v_2 \sim U\{0, 1, 2\}$.
Selling separately revenue:

Bundle revenue:

Different option: One item at 2, both at 3. Revenue:

Example 3: Randomization is Necessary. Suppose $v_1, v_2 \sim F$ where $F = \begin{cases} 1 & \text{w.p. } \frac{1}{6} \\ 2 & \text{w.p. } \frac{1}{2} \\ 4 & \text{w.p. } \frac{1}{3}. \end{cases}$

Selling separately revenue:

Bundle revenue:

Randomized option:

- Pay 1 for a lottery ticket that gets the first item w.p. .5.
- Pay 1 for a lottery ticket that gets the first item w.p. .5.
- Pay 4 to get both items with certainty.

This auction yields revenue $3\frac{17}{36}$.

Example 4: Sometimes Selling Separately is Best. Suppose we have m items where

$$v_i = \begin{cases} 2^i & \text{w.p. } 2^{-i} \\ 0 & \text{otherwise.} \end{cases}$$

Selling separately revenue:

Grand bundle revenue:

Approximately Optimal Revenue for an Additive Buyer

Note: Refer to Lecture 14 from October 25.

Bounding OPT. Previously, we used the Lagrangian duality framework of CDW '16, weak duality, and the Myersonian-like flow of dual variables to achieve the following upper bound on optimal revenue:

$$\begin{aligned} \textsc{Rev}(F) \leq & \sum_{v \in V} \sum_j f(v) \cdot \hat{x}_j(v) \cdot v_j \cdot \mathbb{1}[v \notin R_j] \quad (\textsc{Non-Favorite}) \\ & + \sum_{v \in V} \sum_j f(v) \cdot \hat{x}_j(v) \cdot \bar{\varphi}_j(v_j) \cdot \mathbb{1}[v \in R_j] \quad (\textsc{Single}) \end{aligned}$$

where R_j is the set of valuations v under which item j is the bidder's favorite item, breaking ties lexicographically (by smallest item index), and $\bar{\varphi}_j(v_j)$ is the standard ironed Myersonian virtual value for item j.

Bounding Single. We also saw that $\textsc{Single} \leq \textsc{opt}^{\textsc{Copies}}$ where the $\textsc{Copies}$ setting is a *single-dimensional* setting with nm single-dimensional bidders, where copy (i,j)'s value for winning is v_{ij} (just one parameter—which is still drawn from F_{ij}). $\textsc{opt}^{\textsc{Copies}}(F)$ is the revenue of Myerson's optimal auction in the copies setting induced by F.

The Core-Tail Split of Non-Favorite. When the bidder is additive, we further decompose non-favorite into two terms we call core and tail by partitioning the valuations by those with $v_j \leq r$ and those with $v_j > r$ for some threshold r, where we will choose r to be the optimal revenue earned by posting a separate price on each item, $r = \textsc{srev}$.

$$
\begin{aligned}
&\sum_{v \in V} \sum_j f(v) \cdot \hat{x}_j(v) \cdot v_j \cdot \mathbb{1}[v \notin R_j] \quad (\textsc{Non-Favorite}) \\
&\quad \leq \sum_{v \in V} \sum_j f(v) \cdot v_j \cdot \mathbb{1}[v \notin R_j] \\
&\quad = \sum_j \sum_{v_j > r} f_j(v_j) \cdot v_j \cdot \sum_{v_{-j}} f_{-j}(v_{-j}) \cdot \mathbb{1}[v \notin R_j] + \sum_j \sum_{v_j \leq r} f_j(v_j) \cdot v_j \cdot \sum_{v_{-j}} f_{-j}(v_{-j}) \cdot \mathbb{1}[v \notin R_j] \\
&\quad \leq \sum_j \sum_{v_j > r} f_j(v_j) \cdot v_j \cdot \Pr_{v_{-j}}[v \notin R_j] \quad (\textsc{Tail}) \quad + \sum_j \sum_{v_j \leq r} f_j(v_j) \cdot v_j \quad (\textsc{Core})
\end{aligned}
$$

Bounding the Tail.

Lemma 1. $\textsc{Tail} \leq \textsc{srev}$.

Proof. By the definition of R_j, for any given v_j,

$$\Pr_{v_{-j}}[v \notin R_j] =^{1} \qquad\qquad .$$

Posting a price of v_j on each item separately earns revenue at least
because

then for all v_j,

$$v_j \cdot \Pr_{v_{-j}}[v \notin R_j] \leq$$

Thus

$$
\begin{aligned}
\textsc{Tail} &\leq \\
&= \\
&\leq \textsc{srev}.
\end{aligned}
$$

□

Bounding the Core.

Lemma 2. *If we sell the grand bundle at price* $\textsc{Core} - 2r$*, the bidder will purchase it with probability at least* $1/2$*. Then* $\textsc{brev} \geq \frac{\textsc{Core}}{2} - r$*, or* $\textsc{Core} \leq 2\textsc{brev} + 2\textsc{srev}$.

To prove this, we need the following fact from a technical lemma in CDW.

[1] Really, $\leq$ because of tie-breaking, but that's the right direction.

Technical Fact (CDW '16). Let x be a positive single dimensional random variable drawn from F of finite support such that for any number a, $a \cdot \Pr_{x \sim F}[x \geq a] \leq \mathcal{B}$ where $\mathcal{B}$ is an absolute constant. Then for any positive number s, the second moment of the random variable $x_s = x \cdot \mathbb{1}[x \leq s]$ is upper-bounded by $2\mathcal{B} \cdot s$.

Proof. For each item j define a new random variable c_j that 0's out the part of the distribution not in the core as follows: Draw a sample v_j. If $v_j \in [0, r]$, then $c_j = v_j$. Otherwise, $c_j = 0$.

Let $c = \sum_j c_j$ be the sum of these new core random variables. Notice that $\mathbb{E}[c] = \sum_j \sum_{v_j \leq r} f_j(v_j) \cdot v_j$. We now show that c, the sum of item values drawn only from the core, concentrates, because it has small variance.

We wish to show that pricing the grand bundle at $\mathbb{E}[c] - 2r$ sells with probability at least 1/2, that is, that

$$\Pr_{v \sim F}\left[\sum_j v_j \geq \mathbb{E}[c] - 2r\right] \geq \frac{1}{2}.$$

You may wish to use the following:

- $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
- If $Z = \sum_i X_i$ and the X_i's are drawn independently, then $\mathrm{Var}[Z] = \sum_i \mathrm{Var}[X_i]$.
- Chebyshev's inequality: $\Pr[|X - \mathbb{E}[X]| \geq t] \leq \mathrm{Var}[X]/t^2$ where $t > 0$.
- The above technical fact.
- Let $r_j = \max_x\{x \cdot \Pr_{v_j}[v_j \geq x]\}$ be the optimal selling-separately revenue from item j, and $r = \sum_j r_j$.

□

Putting it All Together.

Theorem 1. *For a single additive bidder,* $\textsc{opt} \leq 2\textsc{brev} + 4\textsc{srev}$.

Proof.