Beyond Dominant-Strategy: Bayesian Settings

There are many reasons why we can't always require dominant strategies when design mechanisms.

(1) Requiring such a strong concept might not be tractable.

(2) Agents do not always have dominant strategies! What then?

We'll now introduce the Bayesian setting.

Suppose the valuation v_i of bidder i is drawn from a prior distribution F_i.

- We abuse notation and let F_i denote the cumulative distribution function (CDF) of the distribution; that is, $F_i(x) = \Pr_{v_i \sim F_i}[v_i \leq x]$.

- We use $f_i(x)$ to denote the probability density function (pdf) of the distribution; that is, $f_i(x) = \frac{d}{dx} F_i(x)$.

- We use $\mathbf{F}$ or $\vec{F}$ to denote the *joint distribution* of the marginal buyer distributions F_i. That is, if the buyers are independently distributed, then $\mathbf{F}$ is the product distribution $\mathbf{F} = \times_i F_i$. (Note however that buyers are not necessarily independently distributed in all settings.)

Unless otherwise noted, we assume that the prior distribution $\mathbf{F}$ is *common knowledge* to all bidders and the mechanism designer (the seller).

Definition 1. A *Bayes-Nash equilibrium (BNE)* for a joint distribution $\mathbf{F}$ is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for all i and v, $\sigma_i(v_i)$ is a best-response when other agents play $\sigma_{-i}(\mathbf{v}_{-i})$ when $\mathbf{v}_{-i} \sim \mathbf{F}_{-i} \mid_{v_i}$.

Claim 1. Consider two identically and independently drawn bidders from $F = U[0,1]$. It is a (symmetric) BNE for each bidder to bid $\sigma_i(v_i) = v_i/2$ in the first-price auction.

Proof. Suppose bidder 2 is playing the strategy $\sigma_2(v_2) = v_2/2$. We prove that bidder 1's best-response is $\sigma_1(v_1) = v_1/2$. For now, call the bid given by $\sigma_1(v_1) = b_1$.

Given bidder 2's strategy, bidder 1's expected utility is

$$
\begin{aligned}
\mathbb{E}_{v_2}[u_i(\sigma_1(v_1), \sigma_2(v_2))] &= v_1 \cdot \mathbb{E}_{v_2}[x_1(\sigma_1(v_1), \sigma_2(v_2))] - \mathbb{E}_{v_2}[p_1(\sigma_1(v_1), \sigma_2(v_2))] \\
&= v_1 \cdot \Pr_{v_2}[b_1 > v_2/2] - \mathbb{E}_{v_2}[b_1 \cdot \mathbb{1}[b_1 > v_2/2]] && x, p \text{ in FPA} \\
&= \Pr_{v_2}[b_1 > v_2/2] \cdot [v_1 - b_1] && \text{def } \mathbb{E} \\
&= F(2b_1) \cdot [v_1 - b_1] && \text{def } F(\cdot) \\
&= \min\{2b_1, 1\} \cdot [v_1 - b_1] && F = U[0,1] \\
&= 2b_1 v_1 - 2b_1^2 \quad \text{if } b_1 \leq 1/2 \\
\frac{d}{db_1} \mathbb{E}_{v_2}[u_i(b_1, v_2/2)] &= 2v_1 - 4b_1 \quad \text{if } b_1 \leq 1/2 && \text{differentiate to max} \\
\implies b_1 &= v_1/2 \quad \text{which is } \leq 1/2 \text{ as } v_1 \leq 1
\end{aligned}
$$

Hence bidder 1's best-response strategy is to bid $\sigma_1(v_1) = v_1/2$ in response to $\sigma_2(v_2) = v_2/2$, and thus these strategies are a BNE. $\qquad\square$

Exercises (optional):

- Extend Myerson's Lemma and the payment identity for Bayesian Incentive-Compatible (BIC) mechanisms.

- Extend the Revelation Principle for BIC mechanisms.

- If you're rusty with probability, show that for $F = U[0,1]$, the expected max of two draws from F is $2/3$ and the expected min is $1/3$.

Theorem 1 (Revenue Equivalence). *The payment rule and revenue of a mechanism is uniquely determined by its allocation. Hence, any two mechanisms with the same allocation must earn the same expected revenue in Bayes-Nash Equilibrium.*

What is this theorem a corollary of? Prove this for the first-price auction and the Vickrey (second-price) auction in the above setting!

Proof. This is just a corollary of Myerson's Lemma! As we pointed out, the *only* variables in the payment identity are the allocation rule! Payment is 100% determined by the allocation rule! Then two mechanisms with the same allocation *must* have the same payments.

Consider the first-price auction and the second-price auction each with two bidders i.i.d. from $U[0,1]$. Let V^1 and V^2 denote the random variables that are the highest and second-highest draws from $U[0,1]$, respectively. Note that two draws from the uniform distribution evenly divide the interval in expectation: $\mathbb{E}_{V^1, V^2 \sim U[0,1]}[V^1] = 2/3$ and $\mathbb{E}_{V^1, V^2 \sim U[0,1]}[V^2] = 1/3$.

In the first-price auction, the item is allocated to V^1 at a payment of its BNE bid of $V^1/2$. Then the expected winner's payment (and thus revenue) is $\frac{1}{2}\mathbb{E}[V^1] = 1/3$.

In the second-price auction, the item is allocated to V^1 at a payment of the second-highest bid $b^2 = V^2$, since Vickrey is DSIC. Then the expected winner's payment (and thus revenue) is $\mathbb{E}[V^2] = 1/3$. $\qquad\square$

Bayesian Settings

Using notions from the Bayesian setting and how bidders Bayesian update as they learn information, we define three stages of the auction:

1. *ex ante*: Before any information has been drawn; i only knows $\mathbf{F}$.

2. *interim*: Values v_i have been drawn; i only knows their own valuation, and thus the updated prior $\mathbf{F}\mid_{v_i}$.

3. *ex post*: The auction has run and concluded. All bidders know all $v_1, \ldots, v_n$.

Typically we discuss the *ex post* allocation and payment rules as a function of all of the values. However, in the Bayesian setting, to reason about BIC, it often makes sense to take in terms of *interim* allocation and payment rules which have the same information as bidder i before the auction is run.

Definition 2. We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given i's valuation:

$$x_i(v_i) = \Pr_{\mathbf{F}}[x_i(\mathbf{v}) = 1 \mid v_i] = \mathbb{E}_{\mathbf{F}}[x_i(\mathbf{v}) \mid v_i]$$

and

$$p_i(v_i) = \mathbb{E}_{\mathbf{F}}[p_i(\mathbf{v}) \mid v_i].$$

Our definition of Bayesian Incentive-Compatibility then follows:

Definition 3. A mechanism with *interim* allocation rule x and *interim* payment rule p is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

Virtual Welfare

Imagine a single buyer will arrive with their private value v. We want to design DSIC mechanisms.

What mechanism should you use to maximize *welfare* $(\sum_i v_i x_i)$ Always give the bidder the item, always give it away for free!

What should you do to maximize (expected) revenue? Post a price that maximizes $\text{REV} = \max_r r \cdot [1 - F(r)]$.

Definition 4. In a deterministic mechanism, given other bids $\mathbf{b}_{-i}$, bidder i's *critical bid* is the minimum bid $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$ such that bidder i is allocated to.

Then with $\mathbf{b}_{-i}$ fixed, for all winning $v_i \geq b_i^*$, i's payment $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$ is their critical bid.

What is winner i's critical bid in a single-item auction? The second-highest bid!

What about in the k identical item setting? The $k + 1^{\text{st}}$ bid!

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References

[1] Jason D. Hartline. Mechanism design and approximation. *Book draft. October*, 122, 2013.

[2] Tim Roughgarden. *Twenty lectures on algorithmic game theory*. Cambridge University Press, 2016.