Revenue Maximization and Myersonian Virtual Welfare

Recap: For a single buyer will arrive with their private value v, for DSIC mechanisms:

- Maximize welfare ($\sum_i v_i x_i$): Always give the bidder the item, always give it away for free!
- Maximize revenue: Post a price that maximizes $\textsc{Rev} = \max_r r \cdot [1 - F(r)]$.
- Critical bid: For a deterministic mechanism, given other bids $\mathbf{b}_{-i}$, bidder i's *critical bid* is the minimum bid $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$ such that bidder i is allocated to. Then with $\mathbf{b}_{-i}$ fixed, for all winning $v_i \geq b_i^*$, i's payment $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$ is their critical bid.
- The revelation principle says that it's without loss to focus only on truthful mechanisms.
- Payment is determined by the allocation:

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz$$

We want to maximize $\mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})]$.

$$\begin{aligned}
\mathbb{E}_{v_i\sim F_i}[p_i(v_i, \mathbf{v}_{-i})] &= \int_0^\infty f_i(v_i) p_i(v_i, \mathbf{v}_{-i})\, dv_i \\
&= \int_0^\infty f_i(v_i) \left[v_i \cdot x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, dz \right] dv_i \\
&= \int_0^\infty \left[f_i(v_i) v_i x_i(v_i, \mathbf{v}_{-i}) - x_i(v_i, \mathbf{v}_{-i}) \left[\int_{v_i}^\infty f_i(z)\, dz \right] \right] dv_i \qquad (*) \\
&= \int_0^\infty \left[f_i(v_i) v_i x_i(v_i, \mathbf{v}_{-i}) - x_i(v_i, \mathbf{v}_{-i}) [1 - F_i(v_i)] \right] dv_i \\
&= \int_0^\infty f_i(v_i) x_i(v_i, \mathbf{v}_{-i}) \left[v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)} \right] dv_i \\
&= \mathbb{E}_{v_i\sim F_i}[\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})]
\end{aligned}$$

where

$$\varphi_i(v_i) = v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}$$

is the Myersonian virtual value and $(*)$ follows by switching the order of integration. Then

$$\textsc{Revenue} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})] = \sum_i \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[p_i(\mathbf{v})] = \sum_i \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})]$$

Note that this does require takes $\mathbb{E}_{\mathbf{v}_{-i}\sim\mathbf{F}_{-i}}$ of both sides of our previous equation.

$$= \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i \varphi_i(v_i)x_i(\mathbf{v})] = \text{Virtual Welfare}$$

Given this conclusion, how should we design our allocation rule x to maximize expected virtual welfare (expected revenue)? Give the item to the bidder with the highest *virtual* value!

When would this cause a problem with incentive-compatibility? When the corresponding x isn't monotone!

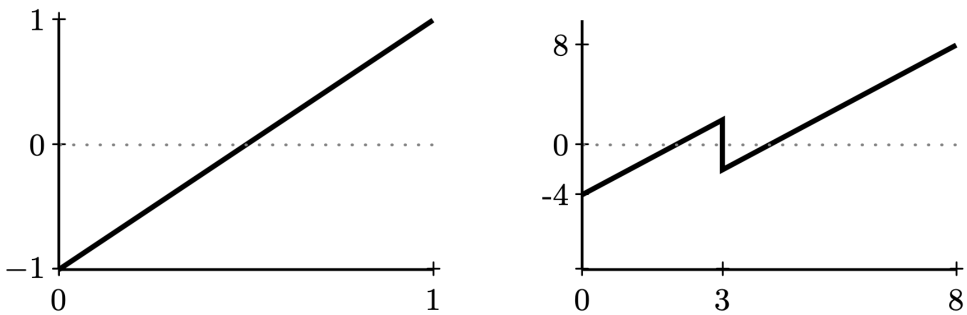
Definition 1. A distribution F is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

Suppose we are in the single-item setting and all of the distributions are regular. What do the payments look like in the virtual-welfare-maximizing allocation?

For a fixed $\mathbf{b}_{-i}$, if i is the winner, then i's payment is i's critical bid, which is $\varphi_i^{-1}(b_2)$ where b_2 is the second highest bid. Exercise: what about for k identical items?

Claim 1. A virtual welfare maximizing allocation x is monotone if and only if the virtual value functions are regular.

Exercise: Argue this.



(a) Uniform agent virtual value.   (b) Bimodal agent virtual value.

Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

It will be helpful to keep the following two examples in mind:

a. a uniform agent with $v \sim U[0,1]$. Then $F(x) = x$ and $f(x) = 1$.

b. a bimodal agent with

$$v \sim \begin{cases} U[0,3] & w.p. \frac{3}{4} \\ U(3,8] & w.p. \frac{1}{4} \end{cases} \quad \text{and} \quad f(v) = \begin{cases} \frac{3}{4} & v \in [0,3] \\ \frac{1}{20} & v \in (3,8] \end{cases}$$

Do the following:

- Calculate the virtual values for both examples.

  a. $\varphi(v) = 2v - 1$

  b. $1 - F(v) = \begin{cases} \frac{1}{4} + \left(\frac{3-v}{3}\right) \cdot \frac{3}{4} & v \in [0,3] \\ \left(\frac{8-v}{5}\right) \cdot \frac{1}{4} & v \in (3,8] \end{cases}$ so $\varphi(v) = \begin{cases} \frac{4}{3}(v-1) & v \in [0,3] \\ 2v - 8 & v \in (3,8] \end{cases}$

- Are they regular? Are there any issues using the allocation that maximizes expected virtual welfare?

  a. Yep!

  b. Nope. As we can see in Figure 1, $\varphi(3.5) = -1 < \varphi(2) = \frac{4}{3}$. This implies a bidder gets allocated with $v = 2$ but then stops getting allocated as they increase their value to 3.5.

- What does that allocation actually look like?

  a. Allocate to all bidders above $v = 0.5$ at a price (critical bid) of $\varphi^{-1}(0) = 0.5$.

  b. The virtual welfare maximizing allocation isn't DSIC! Turns out you can do something to make φ monotone and *then* use the VW-maximizing allocation. We'll do this later in class.

Quantile Space

In *value space*:

- an agent has value v.
- the fraction of the distribution with value above v is $1 - F(v)$.
- the revenue from posting a "take-it-or-leave-it" price of v is $v[1 - F(v)]$.

In *quantile space*: $q = 1 - F(v)$.

- an agent has value v.
- the fraction of the distribution with value above v is $q(v) = 1 - F(q)$.
- the revenue from posting a "take-it-or-leave-it" price of $v(q) = F^{-1}(1 - v)$ is $v(q) \cdot q$.

Example: Consider a distribution that is $U[\$0, \$10]$. Then the quantile 0.1 corresponds to \$9, where 10% of the population might have a higher value. We let $v(q)$ denote the corresponding value, so $v(0.1)$ is \$9.

Definition 2. The *quantile* of a single-dimensional agent with value $v \sim F$ is the measure with respect to F of stronger values, i.e., $q = 1 - F(v)$; the inverse demand curve maps an agent's quantile to her value, i.e., $v(q) = F^{-1}(1 - q)$.

Quantile Distribution: Quantiles are particularly useful because we can draw an agent from any distribution by drawing a quantile $q \sim U[0,1]$. That is, for any $\hat{q}$ and any distribution F, $\Pr_F[q \leq \hat{q}] = \hat{q}$. In English: the probability that an agent has a value in the top 0.3 of the distribution is 0.3.

Note: For everything we do today, we *could* stay in value space, (and sometimes we'll compare), but we'd have to normalize by the distribution using $f(v)$, which makes everything a bit messier and a bit trickier.

Example: For the example of a uniform agent where $F(z) = z$, the inverse demand curve is $v(q) = 1 - q$.

For an allocation rule $x(\cdot)$ in value space, we define an allocation rule in *quantile space* $y(\cdot)$:

$$y(q) = x(v(q)).$$

As $x(\cdot)$ is monotone weakly increasing, then $y(\cdot)$ is monotone *weakly decreasing*.

Definition 3. The *revenue curve* of a single-dimensional agent specified by $R(v) = v \cdot [1 - F(v)]$.

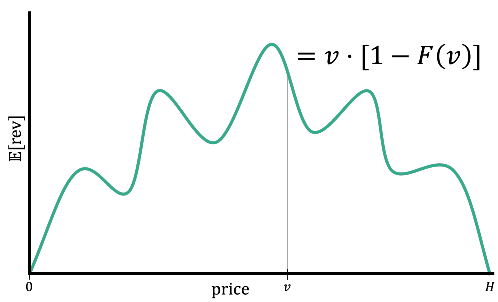


Figure 2: A revenue curve in *value space*.

Note: This is *only* the revenue that can be achieved by posting a single take-it-or-leave-it price. This does not capture the expected revenue of any given mechanism.

Definition 4. The *revenue curve* of a single-dimensional linear agent specified by inverse demand curve $v(\cdot)$ is $P(q) = q \cdot v(q)$ for any $q \in [0,1]$.

Assuming the lower-end of the support of F is 0 and the upper end is some finite $v_{\max}$, then $P(0) = 0$ and $P(1) = 0$.

Claim 2. Any allocation rule $y(\cdot)$ can be expressed as a distribution of posted prices.

Proof. Given the allocation rule $y(\cdot)$, consider the distribution $G^y(z) := 1 - y(z)$. We show that the mechanism that randomly draws a quantile $\hat{q} \sim G^y$ from the distribution G^y and posts the price $v(\hat{q})$ is equivalent.

For a random price $v(\hat{q})$ and fixed quantile q, then

$$\Pr_{\hat{q} \sim G^y}[v(\hat{q}) < v(q)] = \Pr_{\hat{q} \sim G^y}[\hat{q} > q] = 1 - G^y(q) = y(q).$$

□

Claim 3. Any DSIC allocation rule $x(\cdot)$ can be expressed as a distribution of posted prices.

See Figure for an example. In general, the PDF of the distribution of randomized prices is $x'(v)$ for a price of v to achieve an allocation rule of v.

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References

[1] Jason D. Hartline. Mechanism design and approximation. *Book draft. October*, 122, 2013.

[2] Tim Roughgarden. *Twenty lectures on algorithmic game theory.* Cambridge University Press, 2016.