DS 574 Algorithmic Mechanism Design	**Lecture #7**
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Interdependent Values II

What allocation will maximize social welfare?

The Vickrey Auction modified for the interdependent setting: collect signals, compute values, and allocate to the buyer with the highest value.

Payments. What are the payments?

Fix the signals of other bidders $\mathbf{s}_{-i}$. When bidder i is the winner, they pay their value at their *critical signal* s_i^*. That is, at the signal $s_i^*(\mathbf{s}_{-i})$ where they begin winning (being the highest bidder), $s_i^* = \min\{z \mid x_i(v_i(z, \mathbf{s}_{-i})) = 1\}$.

Truthfulness. Is this mechanism EPIC? When might it not be?

When $x_i(\cdot)$ is not monotone in s_i for some i—when, for some $\mathbf{s}_{-i}$, as i increases their signal, they go from being the highest bidder to not the highest bidder.

Assumptions. What assumption could we place on the class of valuations to ensure that the mechanism is always EPIC?

Once i is the highest-valued bidder, then as they increase their signal s_i, they remain the highest bidder. We call this *single-crossing*—once they *cross* the other bidders' values and become the highest, they never cross back down to become lower than the highest. This is *precisely* the condition that makes Vickrey truthful.

More specifically, we will ask that bidder i is most sensitive to their own signal. Formally, the *single-crossing condition* requires that for all bidders i and j,

$$\frac{\partial}{\partial s_i} v_i(s_i, \mathbf{s}_{-i}) \geq \frac{\partial}{\partial s_i} v_j(s_i, \mathbf{s}_{-i}).$$

Beyond Single-Crossing [1]

What happens when we don't have single-crossing? Can we at least guarantee some approximation to social welfare?

Example. [Impossibility for deterministic prior-free mechanisms without SC.] Consider a scenario with two bidders (bidder 1 and bidder 2), where $S_1 = \{0, 1\}$ and $S_2 = \{0\}$, and the following valuation functions:

$$\begin{aligned} v_1(s_1 = 0, s_2 = 0) &= r; & v_1(s_1 = 1, s_2 = 0) &= r; \\ v_2(s_1 = 0, s_2 = 0) &= 1; & v_2(s_1 = 1, s_2 = 0) &= r^2. \end{aligned}$$

It is easy to see that v_1 does not satisfy single-crossing since when s_1 increases, v_1 does not increase but v_2 increases by $r^2 - 1$, making v_1 go from being r times greater than v_2 to being r times smaller than it.

We claim that, for these valuations, no truthful, deterministic, and prior-free mechanism has an approximation ratio better than r. To see this, consider the signal profile $(s_1 = 0, s_2 = 0)$. To get a better than r-approximation for this profile, bidder 1 must win the item. Truthfulness requires the allocation to be monotone in each bidder's signal, hence bidder 1 must also win at report $(s_1 = 1, s_2 = 0)$, which results in an allocation that is a factor of r off from the optimal allocation. Since r is arbitrary, the approximation ratio is arbitrarily bad.

Example. [Impossibility result for randomized mechanisms without SC.] Consider the case where every bidder has the following signal distribution for some small $\varepsilon > 0$,

$$s_i = \begin{cases} 1 & w.p.\ \varepsilon \\ 0 & w.p.\ 1 - \varepsilon, \end{cases}$$

and each agent i has a valuation $v_i(\mathbf{s}) = \prod_{j \neq i} s_j$; that is, the bidder has a value 1 if and only if every other agent has signal 1. The optimal expected welfare is 1 whenever at least $n-1$ bidders have a 1 signal. This happens with probability $\varepsilon^n + n \cdot \varepsilon^{n-1}(1-\varepsilon)$. Therefore,

$$\text{OPT} \quad = \quad \varepsilon^n + n \cdot \varepsilon^{n-1}(1-\varepsilon) \quad > \quad n\varepsilon^{n-1}(1-\varepsilon). \tag{1}$$

Consider any truthful mechanism at profile $(s_i = 0, \mathbf{s}_{-i} = \mathbf{1})$. At this profile, the mechanism gets bidder i's value in welfare with probability that he is allocated, $x_i(s_i = 0, \mathbf{s}_{-i} = \mathbf{1})$, and otherwise gets zero since no other bidder has non-zero value. By monotonicity, for every i, we have that $x_i(s_i = 0, \mathbf{s}_{-i} = \mathbf{1}) \leq x_i(\mathbf{1})$, and by feasibility, $\sum_i x_i(\mathbf{1}) \leq 1$. Under any other profile (where at least two signals are 0), all agents have zero value, so welfare is zero. The expected welfare of any truthful mechanism is thus bounded by

$$\begin{aligned} \text{WELFARE} &= \sum_i \Pr[s_i = 0, \mathbf{s}_{-i} = \mathbf{1}] \cdot x_i(s_i = 0, \mathbf{s}_{-i} = \mathbf{1}) \cdot 1 + \Pr[\mathbf{s} = \mathbf{1}] \sum_i x_i(\mathbf{1}) \cdot 1 \\ &= \sum_i \varepsilon^{n-1}(1-\varepsilon) \cdot x_i(s_i = 0, \mathbf{s}_{-i} = \mathbf{1}) + \varepsilon^n \sum_i x_i(\mathbf{1}) \\ &\leq \varepsilon^{n-1}(1-\varepsilon) \sum_i x_i(\mathbf{1}) + \varepsilon^n \sum_i x_i(\mathbf{1}) \\ &\leq \varepsilon^{n-1}(1-\varepsilon) + \varepsilon^n \\ &= \varepsilon^{n-1}. \end{aligned} \tag{2}$$

Combining (1) with (2), we get that the approximation ratio of any monotone mechanism is $\textsc{Welfare}/\textsc{opt} \leq \frac{1}{n(1-\varepsilon)}$ which can be made arbitrarily close to $1/n$; this is the same as the welfare attained by just allocating to a random bidder.

A Restricted Class. Optimal welfare is not attainable for general valuations. For what *natural* restricted class of valuations can we achieve some α-approximation to optimal social welfare for every profile of signals $\mathbf{s}$ (prior-free) with an EPIC mechanism?

Submodularity over Signals [2]

Definition 1. Valuation $v_i(\cdot)$ is submodular over signals if, for all j, when $\mathbf{s}_{-j}$ is lower, $v_i(\cdot)$ is more sensitive to s_j. For all j, and for any $\mathbf{s}_{-j} \leq \mathbf{s}'_{-j}$:

$$\frac{\partial}{\partial s_j} v_i(s_j, \mathbf{s}_{-j}) \geq \frac{\partial}{\partial s_j} v_i(s_j, \mathbf{s}'_{-j})$$

Random-Sampling Vickrey Auction.

- Elicit s_i from each bidder i.
- Assign each bidder into set A or set B w.p. $1/2$ independently.
- For each bidder $i \in A$, and use proxy value $\hat{v}_i = v_i(s_i, \mathbf{0}_{A\setminus i}, \mathbf{s}_B)$.
- Allocate to the potential winner in A with the highest proxy value.

Theorem 1. *The RS Vickrey Auction is EPIC and achieves a prior-free $\frac{1}{4}$-approximation to the optimal welfare.*

To prove this theorem, we need to address (1) truthfulness and (2) the approximation guarantee.

Truthfulness. Is this allocation monotone? Yes, it is, for each partition!

Approximation. Is $v_i(s_i, \mathbf{0}_{A\setminus i}, \mathbf{s}_B)$ a good way to choose a winner?

Lemma 1 (Key Lemma)**.** *Let v_i be a submodular over signals valuation. Partition all agents other than i uniformly at random into sets A and B. Then*

$$\mathbb{E}_{A,B}[v_i(s_i, \mathbf{0}_A, \mathbf{s}_B)] \geq \frac{1}{2} v_i(\mathbf{s}).$$

Proof. For any $C \subseteq [n] \setminus \{i\}$ and $D = ([n] \setminus \{i\}) \setminus C$, we consider the two events:

- $A = C$ is chosen as the random subset, $B = D$.
- $A = D$ is chosen as the random subset, $B = C$.

First, we show that $v_i(s_i, \mathbf{s}_C, \mathbf{0}_D) + v_i(s_i, \mathbf{0}_C, \mathbf{s}_D) \geq v_i(\mathbf{s})$:

$$\begin{aligned}
v_i(s_i, \mathbf{0}_C, \mathbf{s}_D) &\geq v_i(s_i, \mathbf{0}_C, \mathbf{s}_D) - v_i(s_i, \mathbf{0}_C, \mathbf{0}_D) && \text{by non-negativity of } v_i(\cdot) \\
&\geq v_i(s_i, \mathbf{s}_C, \mathbf{s}_D) - v_i(s_i, \mathbf{s}_C, \mathbf{0}_D) && \text{by submodularity of } v_i(\cdot) \\
&\geq v_i(\mathbf{s}) - v_i(s_i, \mathbf{s}_C, \mathbf{0}_D).
\end{aligned}$$

Now, we conclude by summing over all events (subsets of $[n] \setminus \{i\}$) and coupling them into (C, D) pairs that partition $[n] \setminus \{i\}$, for which the above holds.

Since every item is placed in A or B with equal probability, then each of the 2^{n-1} subsets are selected with equal probability, $1/2^{n-1}$.

$$\begin{aligned}
\mathbb{E}_A\left[v_i(s_i, \mathbf{s}_A, \mathbf{0}_B)\right] &= \sum_{A \subseteq [n] \setminus \{i\}} \Pr[A] \cdot v_i(s_i, \mathbf{s}_A, \mathbf{0}_B) \\
&= \frac{1}{2^{n-1}} \cdot \sum_{A \subseteq [n] \setminus \{i\}} v_i(s_i, \mathbf{s}_A, \mathbf{0}_B) \\
&\geq \frac{1}{2^{n-1}} \cdot \frac{2^{n-1}}{2} v_i(\mathbf{s}) = \frac{1}{2} v_i(\mathbf{s}),
\end{aligned}$$

because there are $2^{n-1}/2$ pairs of subsets that partition $[n] \setminus \{i\}$. $\square$

Proof of Theorem 1. Approximation: Suppose the highest-valued bidder at $\mathbf{s}$ is i^*, so our goal is to approximate $v_{i^*}(\mathbf{s})$: with probability $1/2$, $i^* \in A$, in which case the chosen winner j has true value at least their proxy value which must be at least i^*'s proxy value to be selected the winner.

$$\begin{aligned}
\text{WELFARE} &= \mathbb{E}_{A,B}[v_j(\mathbf{s}) \mid j = \max_{i \in A} \hat{v}_i] \\
&\geq \mathbb{E}_{A,B}[\max_{j \in A} \hat{v}_j] \\
&\geq \frac{1}{2}\, \mathbb{E}_{A,B}[\max_{j \in A} \hat{v}_j \mid i^* \in A] + \frac{1}{2}\, \mathbb{E}_{A,B}[\max_{j \in A} \hat{v}_j \mid i^* \notin A] \\
&\geq \frac{1}{2}\, \mathbb{E}_{A,B}[\hat{v}_{i^*} \mid i^* \in A] + 0 \\
&= \frac{1}{2}\, \mathbb{E}_{A,B}[v_{i^*}(s_{i^*}, \mathbf{0}_{A \setminus i^*}, \mathbf{s}_B) \mid i^* \in A] \\
&\geq \frac{1}{4}\, v_{i^*}(\mathbf{s}). && \text{Key Lemma}
\end{aligned}$$

By the Key Lemma, i^*'s expected proxy value is at least $\frac{1}{2} v_{i^*}(\mathbf{s})$. This gives a $1/4$ approximation.

Truthfulness: If bidder i increases s_i, then their proxy value $v_i(s_i, \mathbf{0}_{A \setminus i}, \mathbf{s}_B)$ increases. $\square$

References

[1] Alon Eden, Michal Feldman, Amos Fiat, and Kira Goldner. Interdependent values without single-crossing. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, EC '18, pages 369–369, New York, NY, USA, 2018. ACM.

[2] Alon Eden, Michal Feldman, Amos Fiat, Kira Goldner, and Anna R. Karlin. Combinatorial auctions with interdependent valuations: Sos to the rescue. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, EC '19, Phoenix, AZ, USA, 2019. ACM.