DS 574 **Prior Independence**
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Robustness: Prior-Independence

Previously, we knew the distribution F and designed an auction for it. What if we do not know F? Values are still drawn from a distribution, and we still compare our revenue with the optimal auction that knows it. The mechanism we run must work without knowing F.

We consider one item and independent values drawn from the same continuous distribution F, with nonnegative values, positive density f, and finite expected value. Assume F is regular: $\varphi(v) = v - (1 - F(v))/f(v)$ is nondecreasing in value. Write $\textsc{opt}(n, F)$ for the optimal expected revenue with n buyers, among truthful, individually rational auctions.

The Bulow–Klemperer Result

One famous result takes the form of resource augmentation.

Theorem 1 (Bulow–Klemperer)**.** *For i.i.d. regular single-item environments,*

$$\textsc{Rev}_{\text{VCG}}(n+1, F) \geq \textsc{opt}(n, F).$$

Vickrey is the second-price auction with no reserve: allocate to a highest bidder and charge the second-highest bid.

One extra independent buyer from the same F replaces tailoring the auction to F. This compares expected revenues in different-sized markets, not payments on each realization. Identical distributions and regularity let us order buyers by either value or virtual value; independence gives the original n buyers their original joint distribution.

The Bulow–Klemperer proof

First, restrict attention to auctions that *must allocate the item.* To maximize expected revenue, maximize expected virtual welfare. If allocation is required, give the item to a buyer with the highest virtual value even when all virtual values are negative. Since the buyers have the same regular distribution, a highest-value buyer also has a highest virtual value. This is Vickrey's allocation rule. With its truthful payments and zero lowest-type utility, it maximizes revenue among truthful, individually rational auctions that always allocate. The revenue identity is an upper bound if a mechanism gives the lowest type positive utility.

Now insert a comparison mechanism M between the optimal auction and Vickrey. Run the optimal auction on the first n buyers. If it leaves the item unsold, give it to buyer $n + 1$ *for free.* This mechanism always allocates. The first n buyers face their original auction; the last buyer's allocation does not depend on their report and they pay zero. Thus M is truthful and individually

rational.

Giving away an unsold item changes no payment, so

$$\textsc{opt}(n, F) = \textsc{Rev}_M(n+1, F) \le \textsc{Rev}_{\text{VCG}}(n+1, F).$$

Only the proof's comparison mechanism needs to know F. The auction we actually run is Vickrey.

Quantile Space

Instead of describing a price by its dollar value, describe how often it sells. Assume F is continuous and strictly increasing on its support. A value v has *tail quantile* $q = 1 - F(v)$: the fraction of the distribution with higher values. Write $v(q) = F^{-1}(1-q)$ for the corresponding value.

For $U[0, 10]$, a value of 9 is in the top 10%, so its quantile is 0.1 and $v(0.1) = 9$. A value of 2 has quantile 0.8. Smaller quantiles correspond to higher values.

A posted price $v(q)$ sells with probability q. Its expected revenue is the *revenue curve*

$$R(q) = qv(q).$$

For example, posting 9 to a buyer drawn from $U[0, 10]$ earns expected revenue $9(0.1) = 0.9$. The height $R(q)$ is an average over the buyer, not the payment whenever a sale occurs.

If $S \sim F$, its quantile $Q = 1 - F(S)$ is uniform on $[0, 1]$. Indeed, $Q \le a$ means the sample is in the top fraction a of values:

$$\Pr[Q \le a] = \Pr[S \ge v(a)] = 1 - F(v(a)) = a.$$

Equal-width intervals in quantile space have equal probability; the corresponding value intervals need not have equal lengths. F still determines $v(q)$ and $R(q)$.

The Single Sample Mechanism

What if we cannot recruit another buyer, but have one independent observation from the same distribution? Let $S \sim F$ be this sample, independent of our buyer's value $V \sim F$. Post price S: collect S if $V \ge S$, and zero otherwise. The sample is not an eligible buyer.

Theorem 2 (Single-sample pricing)**.** *For a continuous regular F as above, posting an independent sample as a price earns at least $\frac{1}{2}\textsc{opt}(1, F)$ in expectation over both the sample and the buyer.*

Proof. First fix the sampled price at s. Expected revenue over the buyer is $s(1 - F(s))$. If this price has quantile q, that is $R(q)$. Now average over the sampled price. Its quantile is uniform, so

$$\textsc{Rev}_{\text{sample}} = \mathbb{E}_S[S(1 - F(S))] = \mathbb{E}_Q[R(Q)] = \int_0^1 R(q)\, dq.$$

This is the area under the revenue curve.

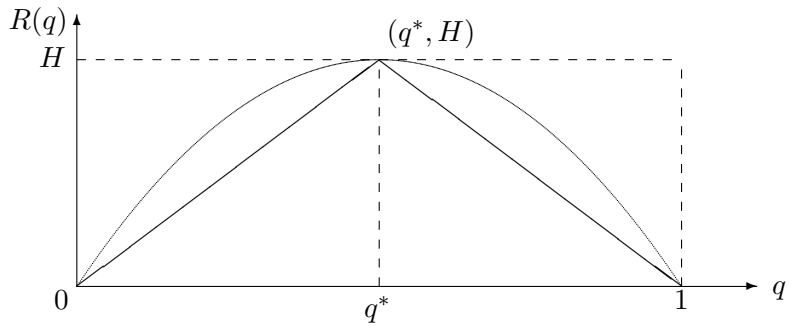
Let q^* maximize $R(q)$, and assume this maximum is attained. The optimal one-buyer auction is a posted price, so $\textsc{opt}(1, F) = R(q^*)$. This is also the area of a rectangle of width 1 and height $R(q^*)$.

Regularity makes R concave. Its endpoint revenues are nonnegative, with $R(0) = 0$ under our finite-mean assumption. Consequently the triangle with vertices $(0, 0)$, $(1, 0)$, and $(q^*, R(q^*))$ lies below the curve. (The actual right endpoint may lie above zero.) The triangle has base 1 and height $R(q^*)$, giving

$$\textsc{Rev}_{\text{sample}} \geq \frac{1}{2} R(q^*) = \frac{1}{2} \textsc{opt}(1, F).$$

The same bound holds for a boundary maximum, using the chord to the other endpoint. The guarantee need not hold for each individual sample.

BK buys optimal revenue with one extra buyer. Single-sample pricing buys half the optimal one-buyer revenue with one independent sample. In both cases, the distribution is used in the analysis but is not needed to run the mechanism.



The rectangle has area $H = \textsc{opt}(1, F)$. The triangle has area $H/2$ and lies beneath the curve. The entire area beneath the curve is the sample mechanism's expected revenue.

Why regularity gives concavity (reference)

Differentiating $q = 1 - F(v(q))$ gives $v'(q) = -1/f(v(q))$, hence

$$R'(q) = v(q) - \frac{q}{f(v(q))} = \varphi(v(q)).$$

As q increases, $v(q)$ decreases, so regularity makes $R'(q)$ nonincreasing. Thus R is concave.

Acknowledgements

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