DS 574 Algorithmic Mechanism Design	**Nash Equilibrium Worksheet**
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Normal-Form Games and Nash Equilibrium

In Golden Balls, each player chooses Split or Steal, and the two choices determine both payoffs. A normal-form game writes down these choices and payoffs for any number of players.

Definition 1. A *normal-form game* consists of a set of players $\{1, \ldots, n\}$; for each player i a set S_i of *strategies*; and for each player i a *payoff function* $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$. A *strategy profile* is a choice $b = (b_1, \ldots, b_n)$ of one strategy per player. We write b_{-i} for everyone's strategy but i's, so that $b = (b_i, b_{-i})$.

The notation b_{-i} says "hold everyone else fixed and vary just player i."

Definition 2. Strategy b_i is a *best response* to b_{-i} if $u_i(b_i, b_{-i}) \geq u_i(b'_i, b_{-i})$ for every $b'_i \in S_i$.

A Nash equilibrium is a profile with *no beneficial unilateral deviation.* That is, nobody can do better by changing only their own strategy.

Definition 3. A strategy profile b is a *pure Nash equilibrium* if every player's strategy is a best response to the others': for every i and every $b'_i \in S_i$,

$$u_i(b_i, b_{-i}) \geq u_i(b'_i, b_{-i}).$$

- A dominant strategy is best against $\qquad\qquad\qquad$ b_{-i}.
- A best response is best against $\qquad\qquad\qquad$ b_{-i}.
- If every player plays a dominant strategy, then

For a matrix game, fix each column and mark all best responses of the row player; then fix each row and mark all best responses of the column player. A cell is a pure Nash equilibrium exactly when both players' choices are marked. Include ties.

	They Split	**They Steal**
You Split	$(J/2,\ J/2)$	$(0,\ J)$
You Steal	$(J,\ 0)$	$(0,\ 0)$

For jackpot $J > 0$, which profiles are pure Nash equilibria? What role does indifference play?

Dominance and Equilibrium

Recall that b_i is *strictly dominated* by c_i if

$$u_i(c_i, b_{-i}) > u_i(b_i, b_{-i}) \qquad \text{for every } b_{-i}.$$

Weak dominance requires $\geq$ for every b_{-i}, with strict inequality for at least one b_{-i}.

Claim 1. No strictly dominated strategy can be played in a pure Nash equilibrium.

Proof.

Does the claim still hold if we replace "strictly dominated" with "weakly dominated"? Use Golden Balls.

A pure Nash equilibrium is *strict* if every change to a different strategy strictly lowers the deviator's payoff. Which Golden Balls equilibria are strict?

Mixed Strategies

In matching pennies, the row player wins 1 if the faces agree and loses 1 otherwise; the column player receives the opposite payoff.

	Column: Heads	**Column: Tails**
Row: Heads	$(1,\ -1)$	$(-1,\ 1)$
Row: Tails	$(-1,\ 1)$	$(1,\ -1)$

Why is there no pure Nash equilibrium?

Definition 4. A *mixed strategy* σ_i is a probability distribution over S_i. Players draw their actions independently. Write

$$U_i(\sigma_i, \sigma_{-i}) = \mathbb{E}_{b \sim \sigma}[u_i(b_i, b_{-i})].$$

A profile σ is a *mixed Nash equilibrium* if, for every player i and every alternative distribution σ_i',

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}).$$

A pure strategy is the special case that puts probability 1 on one action. A mixed deviation changes a player's distribution, holding the opponents' distributions fixed.

Let p be the probability that the row player chooses Heads, and q the probability that the column player chooses Heads. What is the row player's expected payoff from each pure action? For which q can both actions be best responses?

What are the column player's expected payoffs from Heads and Tails? Which p makes the column player indifferent?

For the candidate probabilities you found, verify that neither player can gain by changing to *any* other mixed strategy.

Note that your mixing probability makes your *opponent* indifferent. Indifference identifies the candidate; checking all deviations verifies equilibrium.

Second-Price Auctions

Two bidders have private values $v_1, v_2 \geq 0$ for one item. Each submits a bid $b_i \geq 0$. The highest bidder wins and pays the other bid; ties go to bidder 1. Write $v = (v_1, v_2)$ and $b = (b_1, b_2)$. Let $x_i(b)$ indicate whether bidder i receives the item, and let $p_i(b)$ be its payment:

$$x_1(b) = \begin{cases} 1 & b_1 \geq b_2, \\ 0 & b_1 < b_2, \end{cases} \qquad x_2(b) = 1 - x_1(b), \qquad p_1(b) = b_2 x_1(b), \quad p_2(b) = b_1 x_2(b).$$

Buyer utility. A bidder's utility is its value for the item received minus its payment:

$$u_i(b; v_i) = v_i x_i(b) - p_i(b) = \begin{cases} v_i - b_j & \text{if bidder } i \text{ wins}, \\ 0 & \text{if bidder } i \text{ loses}, \end{cases} \qquad j \neq i.$$

Values are fixed when we check deviations; a bidder changes its bid, not its value.

Welfare. Social welfare is the total value of the allocation:

$$W(b; v) = \sum_{i=1}^{2} v_i x_i(b).$$

Thus welfare is the winner's true value. Assuming the seller has value 0 for the item, this equals the sum of buyer utilities and seller revenue:

$$W(b; v) = \sum_{i=1}^{2} u_i(b; v_i) + \sum_{i=1}^{2} p_i(b).$$

Payments cancel because they are transfers from buyers to the seller.

Exercise. Fix an arbitrary bidder i, its value v_i, and the other bidder's bid B. Show that bidding $b_i = v_i$ is a dominant strategy. What does this imply about the truthful bid profile?

Exercise. Let $v_1 = 10$ and $v_2 = 6$. Is the bid profile $(b_1, b_2) = (5, 12)$ a pure Nash equilibrium? Check every possible unilateral bid change. Does it maximize welfare? How does it fit with our claim about dominated strategies?

Theorem 1 (Nash, 1950)**.** *Every game with finitely many players and finitely many strategies each has at least one Nash equilibrium in mixed strategies.*