DS 574 Algorithmic Mechanism Design	**Nash Equilibrium**
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Normal-Form Games and Nash Equilibrium

In Golden Balls, each player chooses Split or Steal, and the two choices determine both payoffs. A normal-form game writes down these choices and payoffs for any number of players.

Definition 1. A *normal-form game* consists of a set of players $\{1, \ldots, n\}$; for each player i a set S_i of *strategies*; and for each player i a *payoff function* $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$. A *strategy profile* is a choice $b = (b_1, \ldots, b_n)$ of one strategy per player. We write b_{-i} for everyone's strategy but i's, so that $b = (b_i, b_{-i})$.

The notation b_{-i} says "hold everyone else fixed and vary just player i."

Definition 2. Strategy b_i is a *best response* to b_{-i} if $u_i(b_i, b_{-i}) \geq u_i(b'_i, b_{-i})$ for every $b'_i \in S_i$.

A Nash equilibrium is a profile with *no beneficial unilateral deviation.* That is, nobody can do better by changing only their own strategy.

Definition 3. A strategy profile b is a *pure Nash equilibrium* if every player's strategy is a best response to the others': for every i and every $b'_i \in S_i$,

$$u_i(b_i, b_{-i}) \geq u_i(b'_i, b_{-i}).$$

- A dominant strategy is best against *every* b_{-i}.
- A best response is best against *one particular* b_{-i}.
- If every player plays a dominant strategy, then the profile is Nash.

For a matrix game, fix each column and mark all best responses of the row player; then fix each row and mark all best responses of the column player. A cell is a pure Nash equilibrium exactly when both players' choices are marked. Include ties.

	They Split	**They Steal**
You Split	$(J/2,\ J/2)$	$(0,\ J)$
You Steal	$(J,\ 0)$	$(0,\ 0)$

For jackpot $J > 0$, which profiles are pure Nash equilibria? What role does indifference play?

Golden Balls. The pure Nash equilibria are (Steal, Steal), (Split, Steal), and (Steal, Split). Against Split, Steal pays $J > J/2$; against Steal, both choices pay 0. In each asymmetric equilibrium the Splitter is indifferent, so switching does not improve their payoff. (Split, Split) is not Nash: either player can increase their payoff by Stealing.

Dominance and Equilibrium

Recall that b_i is *strictly dominated* by c_i if

$$u_i(c_i, b_{-i}) > u_i(b_i, b_{-i}) \qquad \text{for every } b_{-i}.$$

Weak dominance requires $\geq$ for every b_{-i}, with strict inequality for at least one b_{-i}.

Claim 1. No strictly dominated strategy can be played in a pure Nash equilibrium.

Proof. Fix a pure Nash equilibrium b. Suppose some player's strategy b_i is strictly dominated by c_i. Strict dominance holds against every profile of the other players, so in particular

$$u_i(c_i, b_{-i}) > u_i(b_i, b_{-i}).$$

Then c_i is a beneficial unilateral deviation, contradicting the equilibrium condition. □

Does the claim still hold if we replace "strictly dominated" with "weakly dominated"? Use Golden Balls.

Weak dominance. No! Steal weakly dominates Split: it pays strictly more against Split and the same against Steal. But Split is played in both asymmetric pure Nash equilibria. Against the opponent's equilibrium choice, the dominating strategy need not give a strict improvement.
A pure Nash equilibrium is *strict* if every change to a different strategy strictly lowers the deviator's payoff. Which Golden Balls equilibria are strict?

Strictness. None. In (Steal, Steal) both players are indifferent; in each asymmetric equilibrium the Splitter is indifferent. Strictness requires every unilateral change to be strictly worse.

Mixed Strategies

In matching pennies, the row player wins 1 if the faces agree and loses 1 otherwise; the column player receives the opposite payoff.

	Column: Heads	**Column: Tails**
Row: Heads	$(1, -1)$	$(-1, 1)$
Row: Tails	$(-1, 1)$	$(1, -1)$

Why is there no pure Nash equilibrium?

Pure equilibria. At every profile the losing player can switch faces and raise their payoff from -1 to 1.

Definition 4. A *mixed strategy* σ_i is a probability distribution over S_i. Players draw their actions independently. Write

$$U_i(\sigma_i, \sigma_{-i}) = \mathbb{E}_{b\sim\sigma}[u_i(b_i, b_{-i})].$$

A profile σ is a *mixed Nash equilibrium* if, for every player i and every alternative distribution σ_i',

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}).$$

A pure strategy is the special case that puts probability 1 on one action. A mixed deviation changes a player's distribution, holding the opponents' distributions fixed.

Let p be the probability that the row player chooses Heads, and q the probability that the column player chooses Heads. What is the row player's expected payoff from each pure action? For which q can both actions be best responses?

Row payoffs.

$$U_1(H,q) = q \cdot 1 + (1-q)(-1) = 2q - 1,$$
$$U_1(T,q) = q(-1) + (1-q) \cdot 1 = 1 - 2q.$$

If both actions receive positive probability in a best response, their expected payoffs must be equal—otherwise moving probability to the better action improves utility. Thus

$$2q - 1 = 1 - 2q \quad \Longrightarrow \quad q = \tfrac{1}{2}.$$

What are the column player's expected payoffs from Heads and Tails? Which p makes the column player indifferent?

Column payoffs.

$$U_2(H;p) = p(-1) + (1-p) \cdot 1 = 1 - 2p,$$
$$U_2(T;p) = p \cdot 1 + (1-p)(-1) = 2p - 1.$$

Indifference requires $1 - 2p = 2p - 1$, so $p = \frac{1}{2}$.
For the candidate probabilities you found, verify that neither player can gain by changing to *any* other mixed strategy.

Verification. Fix the column player's distribution $q = 1/2$. Each pure action gives the row player expected payoff 0. Any alternative probability $p' \in [0,1]$ therefore gives

$$U_1(p', \tfrac{1}{2}) = \underbrace{p' \cdot 0 + (1-p') \cdot 0}_{\text{expected payoff of the alternative mixture}} = 0.$$

The row player's equilibrium payoff is also 0, so no deviation improves it. With $p = 1/2$ fixed, the same argument gives $U_2(q'; \frac{1}{2}) = 0$ for every $q' \in [0,1]$. Thus $(p,q) = (1/2, 1/2)$ is a mixed Nash equilibrium.
Note that your mixing probability makes your *opponent* indifferent. Indifference identifies the candidate; checking all deviations verifies equilibrium.

Second-Price Auctions

Two bidders have private values $v_1, v_2 \geq 0$ for one item. Each submits a bid $b_i \geq 0$. The highest bidder wins and pays the other bid; ties go to bidder 1. Write $v = (v_1, v_2)$ and $b = (b_1, b_2)$. Let $x_i(b)$ indicate whether bidder i receives the item, and let $p_i(b)$ be its payment:

$$x_1(b) = \begin{cases} 1 & b_1 \geq b_2, \\ 0 & b_1 < b_2, \end{cases} \qquad x_2(b) = 1 - x_1(b), \qquad p_1(b) = b_2 x_1(b), \quad p_2(b) = b_1 x_2(b).$$

Buyer utility. A bidder's utility is its value for the item received minus its payment:

$$u_i(b; v_i) = v_i x_i(b) - p_i(b) = \begin{cases} v_i - b_j & \text{if bidder } i \text{ wins,} \\ 0 & \text{if bidder } i \text{ loses,} \end{cases} \qquad j \neq i.$$

Values are fixed when we check deviations; a bidder changes its bid, not its value.

Welfare. Social welfare is the total value of the allocation:

$$W(b; v) = \sum_{i=1}^{2} v_i x_i(b).$$

Thus welfare is the winner's true value. Assuming the seller has value 0 for the item, this equals the sum of buyer utilities and seller revenue:

$$W(b; v) = \sum_{i=1}^{2} u_i(b; v_i) + \sum_{i=1}^{2} p_i(b).$$

Payments cancel because they are transfers from buyers to the seller.

> **Exercise.** Fix an arbitrary bidder i, its value v_i, and the other bidder's bid B. Show that bidding $b_i = v_i$ is a dominant strategy. What does this imply about the truthful bid profile?

Truthful bidding. Losing gives 0; winning costs B and gives $v_i - B$, regardless of the winning bid. If $v_i < B$, losing is better and bidding v_i loses. If $v_i > B$, winning is better and bidding v_i wins. If $v_i = B$, both outcomes give 0, so the tie-breaking rule does not matter. Truthful bidding maximizes utility for every B, hence is dominant.

The truthful bid profile is therefore Nash. It awards the item to a bidder with the highest value, achieving the maximum possible welfare $W(v; v) = \max\{v_1, v_2\}$.

> **Exercise.** Let $v_1 = 10$ and $v_2 = 6$. Is the bid profile $(b_1, b_2) = (5, 12)$ a pure Nash equilibrium? Check every possible unilateral bid change. Does it maximize welfare? How does it fit with our claim about dominated strategies?

Bidder 1. Currently loses and gets 0. A bid below 12 still loses; a bid at least 12 wins and gives $10 - 12 = -2$. No deviation improves utility.

Bidder 2. Currently wins, pays 5, and gets $6 - 5 = 1$. A bid above 5 still wins at price 5; a bid at most 5 loses and gives 0. No deviation improves utility, so $(5, 12)$ is Nash.

Welfare and dominance. Welfare is $W((5,12);(10,6)) = 6$, while awarding the item to bidder 1 gives 10. Buyer utilities are 0 and 1, and seller revenue is 5, so total welfare is $0 + 1 + 5 = 6$. Bidder 2's bid 12 is weakly dominated by truthful bidding 6: truth is never worse, and against $B = 8$ it loses for 0 while bidding 12 wins for -2. Against the equilibrium bid $B = 5$, both bids give utility 1. Our exclusion result applies to *strictly* dominated strategies.

Theorem 1 (Nash, 1950)**.** *Every game with finitely many players and finitely many strategies each has at least one Nash equilibrium in mixed strategies.*

We take the existence theorem as given; its proof uses a fixed-point argument. Existence does not guarantee efficiency. The *price of anarchy* compares the worst equilibrium's performance with the optimal outcome.

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