Redistributive Allocation Mechanisms

Work by Mohammad Akbarpour ® Piotr Dworczak ® Scott Kominers

Background

Many goods and services are allocated using non-market mechanisms, even though monetary transfers are available:

- Public housing
- Health care
- Food (e.g., food stamps)
- Vaccines
- Event tickets
- …

Why not use a market mechanism, which could lead to more efficient outcomes? One major concern: **redistributive outcomes**

Redistributive Allocation Mechanism

- Market design framework for optimal (one-sided) allocation under redistributive concerns.
  - Designer allocates goods of **heterogeneous** quality to agents differing in **observed** and **unobserved** characteristics.
  - They derive the optimal mechanism under **IC and IR constraints** and an objective function that reflects redistributive concerns via **welfare weights.**

They show that the optimal mechanism is shaped by:

1. the interaction between social preferences and observability of agents' characteristics;
2. how the revenue generated by the mechanism is used;
3. whether the good is "universally desirable."

Outline

- Model
- Derivation of Optimal Mechanisms
- Economic Implications

Model

A designer chooses a mechanism to allocate a unit mass of objects to a unit mass of agents.

- Each object has quality $q \in [0,1]$, $q \sim F$
- Agent types (i, r, λ), where:
  - i is an observable **label**, $i \in I$ (finite);
  - r is an unobserved **willingness to pay** (for quality), $r \in R^+$;
  - λ is an unobserved **social welfare weight**, $\lambda \in R^+$;

Model

A designer chooses a mechanism to allocate a unit mass of objects to a unit mass of agents.

- Each object has quality $q \in [0,1]$, $q \sim F$
- Agent types (i, r, λ), where:
  - i is an observable **label**, $i \in I$ (finite);
    - Agents with the same label form a **group**; there are $\mu_i > 0$ agents in group i
  - r is an unobserved **willingness to pay** (for quality), $r \in R^+$;
  - λ is an unobserved **social welfare weight**, $\lambda \in R^+$;

Model

A designer chooses a mechanism to allocate a unit mass of objects to a unit mass of agents.

- Each object has quality $q \in [0,1]$, $q \sim F$
- Agent types (i, r, λ), where:
  - i is an observable **label**, $i \in I$ (finite);
    - Agents with the same label form a **group**; there are $\mu_i > 0$ agents in group i
  - r is an unobserved **willingness to pay** (for quality), $r \in R^+$;
    - Conditional on label $i, r \sim G_i$ with continuous density g_i and strictly positive on $[\underline{r}_i, \overline{r}_i]$
  - λ is an unobserved **social welfare weight**, $\lambda \in R^+$;

Model

A designer chooses a mechanism to allocate a unit mass of objects to a unit mass of agents.

- Each object has quality $q \in [0,1]$, $q \sim F$
- Agent types (i, r, λ), where:
  - i is an observable **label**, $i \in I$ (finite);
    - Agents with the same label form a **group**; there are $\mu_i > 0$ agents in group i
  - r is an unobserved **willingness to pay** (for quality), $r \in R^+$;
    - Conditional on label $i, r \sim G_i$ with continuous density g_i and strictly positive on $[\underline{r}_i, \overline{r}_i]$
  - λ is an unobserved **social welfare weight**, $\lambda \in R^+$;
- If (i, r, λ) gets a good with quality q and pays t, her utility is $qr - t$, while her contribution to social welfare is $\lambda(qr - t)$.

Allocations and Mechanisms

An assignment Γ is a collection of $|I|$ measurable functions

$$\Gamma_i : [\underline{r_i}, \overline{r}_i] \rightarrow \Delta(Q) \text{ with } \Gamma_i(q|\, r)$$

The assignment is feasible if:

$$\Gamma_i(\cdot|\, r) \text{ is a CDF for all } i, \text{ and } r \in [\underline{r}_i, \bar{r}_i];$$

$$\sum_{i \in I} \mu_i \int_{\underline{r}_i}^{\bar{r}_i} \Gamma_i(q|\, r) dG_i(r) \geq F(q), \forall q \in Q.$$

Model

The designer has access to arbitrary (direct) allocation mechanisms $(\Gamma_i, t_i)_{\{i \in I\}}$, subject to:

- **IC constraint:** Each agent (i, r, λ) reports (r, λ) truthfully;
- **IR constraint:** $U(i, r, \lambda) \equiv r \int q d\Gamma(q|i, r, \lambda) - T(i, r, \lambda) \geq 0$;
- **Non-negative transfers:** $T(i, r, \lambda) \geq 0, \ \forall(i, r, \lambda)$.

Model

A mechanism is **feasible** if and only if Γ is a feasible assignment, $Q^{\Gamma_i}(r)$ is non-decreasing in r for all i, and $t_i(r)$ satisfies, for some $\underline{U_i} \in \left[0, Q^{\Gamma_i}\left(\underline{r_i}\right)\underline{r_i}\right]$:

$$U_i(r) \equiv rQ^{\Gamma_i}(r) - t_i(r) = \underline{U_i} + \int_{\underline{r_i}}^{r} Q^{\Gamma_i}(\tau)d\tau$$

Where $Q^{\Gamma_i}(r) = \int_0^1 q d\Gamma_i(q|r)$.

Model

The designer has access to arbitrary (direct) allocation mechanisms $(\Gamma_i, t_i)_{\{i \in I\}}$, subject to:

- **IC constraint:** Each agent (i, r, λ) reports (r, λ) truthfully;
- **IR constraint:** $U(i, r, \lambda) \equiv r \int q d\Gamma(q|i, r, \lambda) - t_i(r, \lambda) \geq 0$;
- **Non-negative transfers:** $t_i(r, \lambda) \geq 0, \ \forall (i, r, \lambda)$.

The designer maximizes, **α-weighted sum of revenue and utility:**

$$E_{(i,r,\lambda)}[\alpha t_i(r, \lambda) + \lambda U(i, r, \lambda)] .$$

Comment on the Model

Lemma: It is without loss of optimality for the mechanism designer to only elicit information about r through the mechanism — allocation and transfers depend on (i, r) but not on λ directly.

Comment on the Model

Lemma: It is without loss of optimality for the mechanism designer to only elicit information about r through the mechanism — allocation and transfers depend on (i, r) but not on λ directly.

This implies that designer forms expectations over the unobserved social welfare weights $\lambda_i(r) \equiv \mathbb{E}_{(i,r,\lambda)}[\lambda \mid i, r]$.

Comment on the Model

Lemma: It is without loss of optimality for the mechanism designer to only elicit information about r through the mechanism — allocation and transfers depend on (i, r) but not on λ directly.

This implies that designer forms expectations over the unobserved social welfare weights $\lambda_i(r) \equiv \mathbb{E}_{(i,r,\lambda)}[\lambda \mid i, r]$; they call these functions **Pareto weights**, and assume they are continuous.

Comment on the Model

Lemma: It is without loss of optimality for the mechanism designer to only elicit information about r through the mechanism — allocation and transfers depend on (i, r) but not on λ directly.

This implies that designer forms expectations over the unobserved social welfare weights $\lambda_i(r) \equiv \mathbb{E}_{(i,r,\lambda)}[\lambda \mid i, r]$.

Designer's objective becomes:

$$E_{(i,r)}[\alpha T_i(r) \;+\; \lambda_i(r) U_i(r)]\,.$$

Examples

- If label i captures income brackets, then we may naturally think that
$$\mathbb{E}[\lambda \mid i] \equiv \overline{\lambda_i} \geq \overline{\lambda_j} \equiv \mathbb{E}[\lambda \mid j]$$
if i corresponds to a lower income bracket than j

- Suppose there is just one label $I = \{i\}$, but we elicit information about willingness to pay of two people for a house:

Narun: $50,000

Taylor: $500,000

Examples

- If label i captures income brackets, then we may naturally think that
$$\mathbb{E}[\lambda \mid i] \equiv \overline{\lambda_i} \geq \overline{\lambda_j} \equiv \mathbb{E}[\lambda \mid j]$$
if i corresponds to a lower income bracket than j

- Suppose there is just one label $I = \{i\}$, but we elicit information about willingness to pay of two people for a Taylor Swift concert:

Narun: $1

Taylor: $10

Naturally, we may presume that $\lambda_i(r)$ is nearly **constant** in r.

The objective function

$$\underbrace{\alpha \sum_{i \in I} \mu_i \left(\int_{\underline{r}_i}^{\overline{r}_i} t_i(r) dG_i(r) \right)}_{\text{revenue}} + \underbrace{\sum_{i \in I} \mu_i \left(\int_{\underline{r}_i}^{\overline{r}_i} \lambda_i(r) U_i(r) dG_i(r) \right)}_{\text{social surplus with weights } \lambda_i}$$

The objective function

Let $\overline{\lambda}_i \equiv \int_{\underline{r}_i}^{\overline{r}_i} \lambda_i(\tau) dG_i(\tau)$, $\Lambda_i(r) \equiv \mathbb{E}_{\tilde{r}\sim G_i}[\lambda_i(\tilde{r}) | \tilde{r} \geq r]$,

$$h_i(r) \equiv \frac{1 - G_i(r)}{g_i(r)}, \quad J_i(r) \equiv r - \frac{1 - G_i(r)}{g_i(r)}$$

$$\sum_{i \in I} \mu_i \left(\int_{\underline{r}_i}^{\overline{r}_i} V_i(r) Q^{\Gamma_i}(r) dG_i(r) + \left(\overline{\lambda_i} - \alpha\right) \underline{U_i} \right),$$

where $V_i(r) \equiv \alpha J_i(r) + \Lambda_i(r) h_i(r)$.

The role of revenue

The weight α on revenue captures the value of a dollar in the designer's (unmodeled) budget, spent on the most valuable "cause."

- When $\alpha = \max_i \overline{\lambda}_i$, then it is as if a lump-sum transfer to group i were allowed;
- when $\alpha = \overline{\lambda}$, then it is as if lump-sum transfers to all agents were allowed;
- When $\alpha > \overline{\lambda}_i$ for all i, there is an "outside cause;"
- When $\alpha < \overline{\lambda}_i$ for some i, lump-sum payments to agents in group i are prohibited or costly

Derivation of Optimal Mechanism

The optimal mechanism is found in two steps:

1. The objects are allocated “across” groups: F is split into $|I|$ cdfs F_i^*;
2. The objects are allocated “within” groups: For each label i, the objects F_i^* are allocated optimally to agents in group i.

Derivation of Optimal Mechanism

The optimal mechanism is found in two steps:

1. The objects are allocated "across" groups: F is split into $|I|$ cdfs F_i^*;
2. The objects are allocated "within" groups: For each label i, the objects F_i^* are allocated optimally to agents in group i.

Observation: Only expected quality, $Q_i^*(r)$, matters for payoffs.

Derivation of Optimal Mechanism

The optimal mechanism is found in two steps:

1. The objects are allocated "across" groups: F is split into $|I|$ cdfs F_i^*;
2. The objects are allocated "within" groups: For each label i, the objects F_i^* are allocated optimally to agents in group i.

Observation: Only expected quality, $Q_i^*(r)$, matters for payoffs.

- Assortative matching (market allocation)

$$Q_i^*(r) = (F_i^*)^{-1}\big(G_i(r)\big)\ \forall\ r \in [a, b];$$

- Random matching (non-market allocation)

$$Q_i^*(r) = \overline{q} \ \text{ for some } \overline{q} \text{ and all } r \in [a, b].$$

Setup of Within-Group Allocation

Maximize

$$\sum_{i \in I} \mu_i \left(\int_{\underline{r}_i}^{\overline{r}_i} V_i(r) Q^{\Gamma_i}(r) dG_i(r) + \left(\overline{\lambda_i} - \alpha\right) \underline{U_i} \right)$$

subject to feasibility with $I = \{i\}, \mu_i = 1$, and $F = F_i$.

For a function Ψ, let $co(\Psi)$ denote the concave closure of Ψ and let $cd(\Psi)$ denote the concave decreasing closure of Ψ.

Within-Group Allocation Procedure

1. Compute the function:

$$\Psi_i(t) \equiv \int_t^1 V_i\left(G_i^{-1}(x)\right) dx + \max\{0, \overline{\lambda_i} - \alpha\}\mathbf{1}_{\{t=0\}}$$

Within-Group Allocation Procedure

1. Compute the function:

$$\Psi_i(t) \equiv \int_t^1 V_i\left(G_i^{-1}(x)\right) dx + \max\{0, \overline{\lambda_i} - \alpha\}\mathbf{1}_{\{t=0\}}$$

2. Compute the concave closure $co(\Psi_i)$ and the concave decreasing closure $cd(\Psi_i)$ of Ψ_i.

Within-Group Allocation Procedure

1. Compute the function:

$$\Psi_i(t) \equiv \int_t^1 V_i\left(G_i^{-1}(x)\right) dx + \max\{0, \overline{\lambda_i} - \alpha\}\mathbf{1}_{\{t=0\}}$$

2. Compute the concave closure $co(\Psi_i)$ and the concave decreasing closure $cd(\Psi_i)$ of Ψ_i.
3. If on some initial interval $[0, x_i^*], co(\Psi_i) < cd(\Psi_i)$, then objects of quality below the x_i^* quantile of F_i are not allocated and hence agents with willingness to pay below $r_i^* = G_i^{-1}(x_i^*)$ are assigned quality 0

Within-Group Allocation Procedure

1. Compute the function:

$$\Psi_i(t) \equiv \int_t^1 V_i\left(G_i^{-1}(x)\right) dx + \max\{0, \overline{\lambda_i} - \alpha\}\mathbf{1}_{\{t=0\}}$$

2. Compute the concave closure $co(\Psi_i)$ and the concave decreasing closure $cd(\Psi_i)$ of Ψ_i.
3. If on some initial interval $[0, x_i^*]$, $co(\Psi_i) < cd(\Psi_i)$, then objects of quality below the x_i^* quantile of F_i are not allocated and hence agents with willingness to pay below $r_i^* = G_i^{-1}(x_i^*)$ are assigned quality 0
4. Whenever $co(\Psi_i)$ is affine on a (maximal) interval, the matching between types and quality is random within that interval; whenever $co(\Psi_i)$ is strictly concave on an interval, the matching between types and quality is assortative.

What is this function?

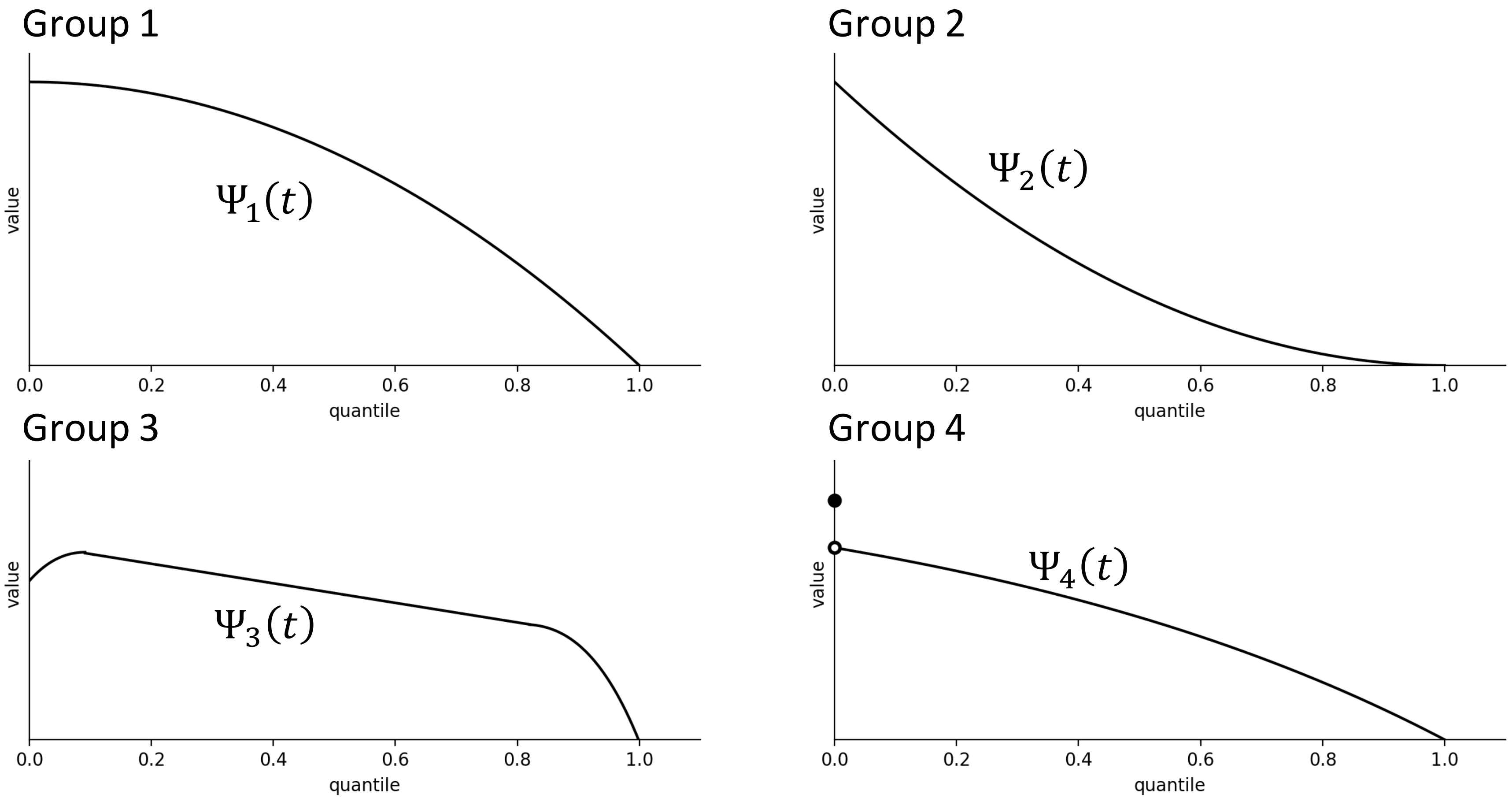
For any $r > \underline{r_i}$,

$$\Psi_i\big(G_i(r)\big) = \int_r^{\overline{r}_i} \tau\lambda_i(\tau)dG_i(\tau) + \big(\alpha - \Lambda_\mathrm{i}(r)\big)r\big(1 - G_i(r)\big).$$

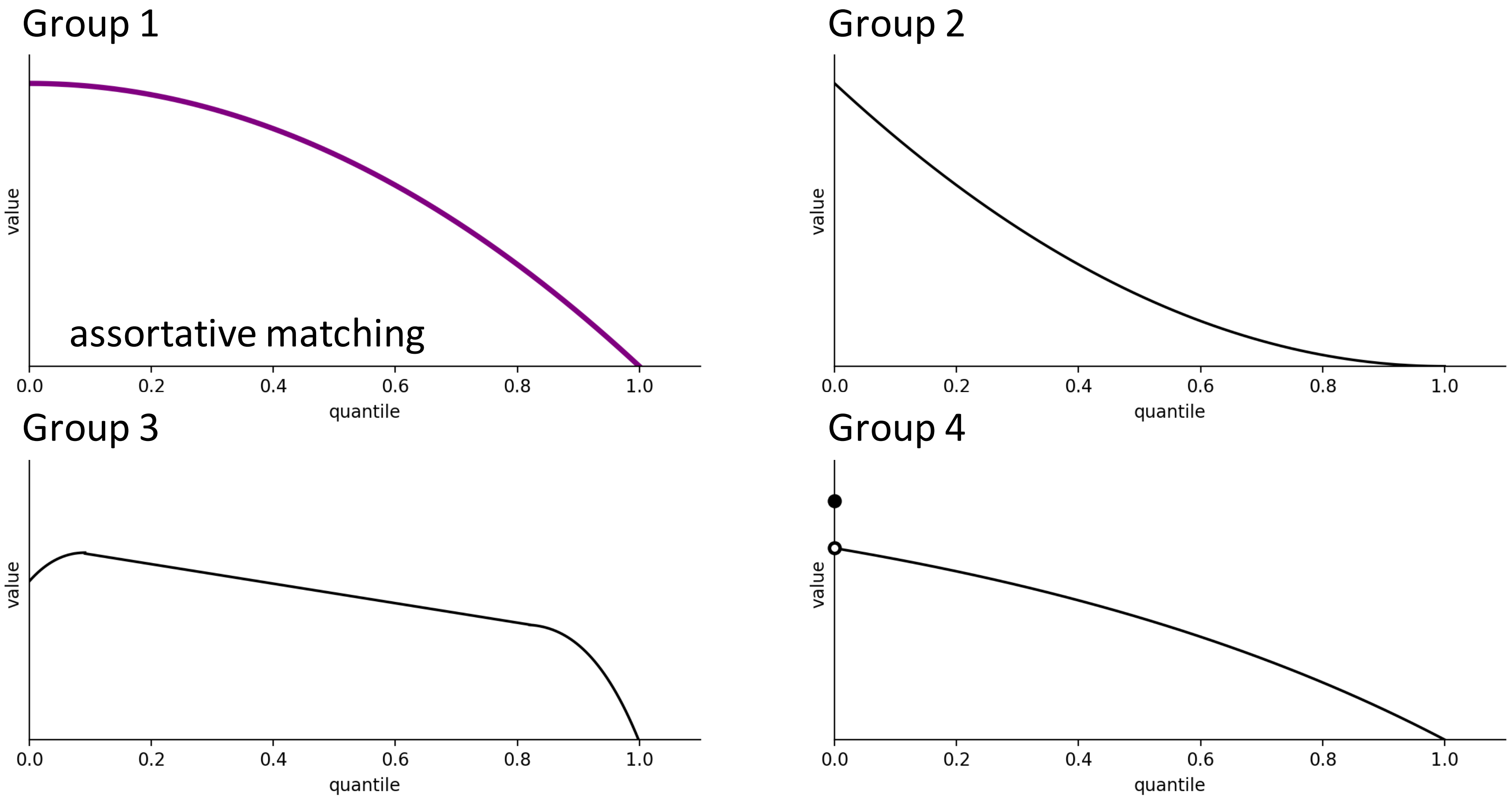
Interpretation:

The value of Ψ_i at some quantile $x = G_i(r)$, is the value to the designer from selling quality 1 at a price of r.

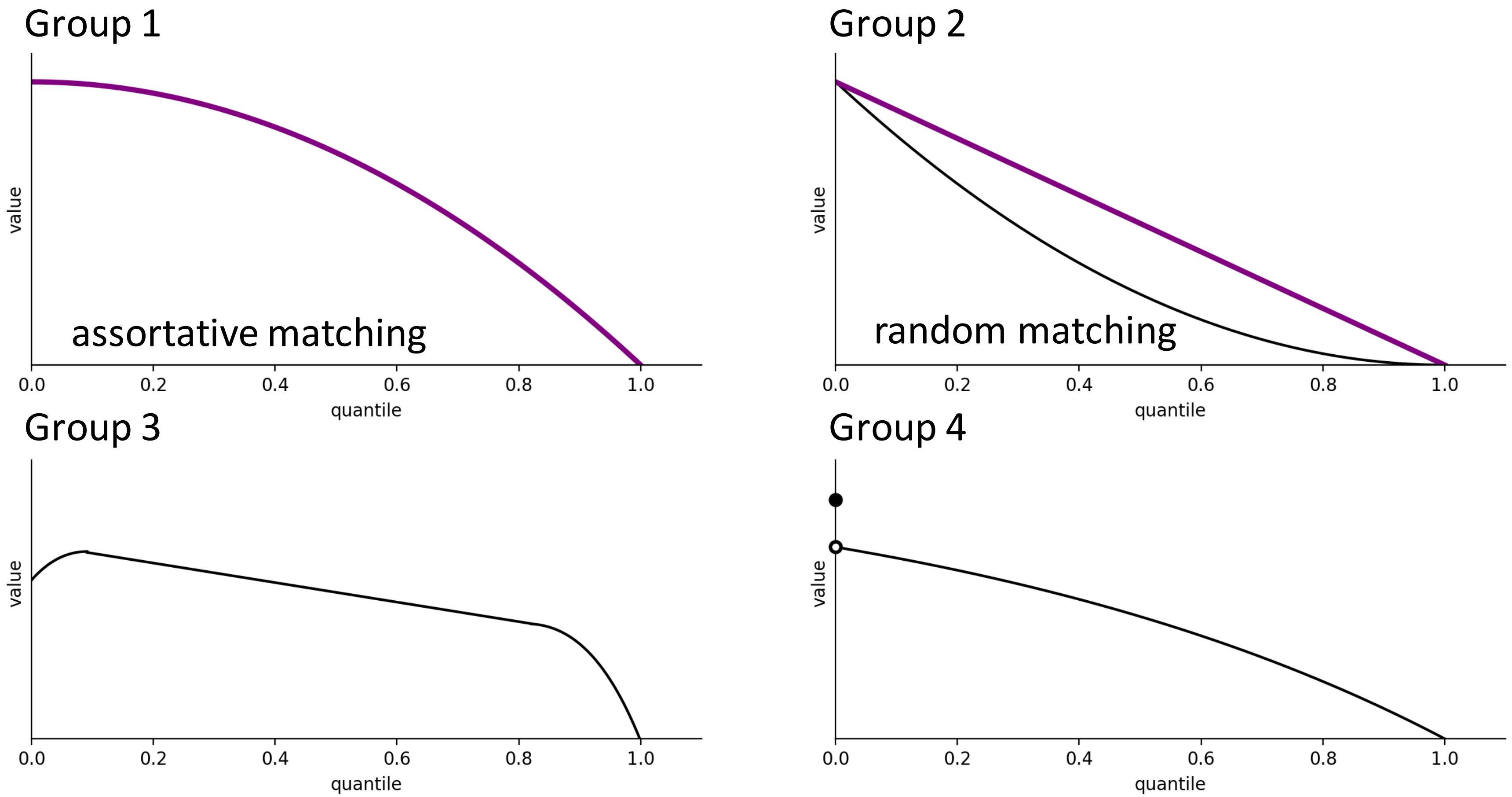
Derivation of Optimal Mechanism: within-group allocation



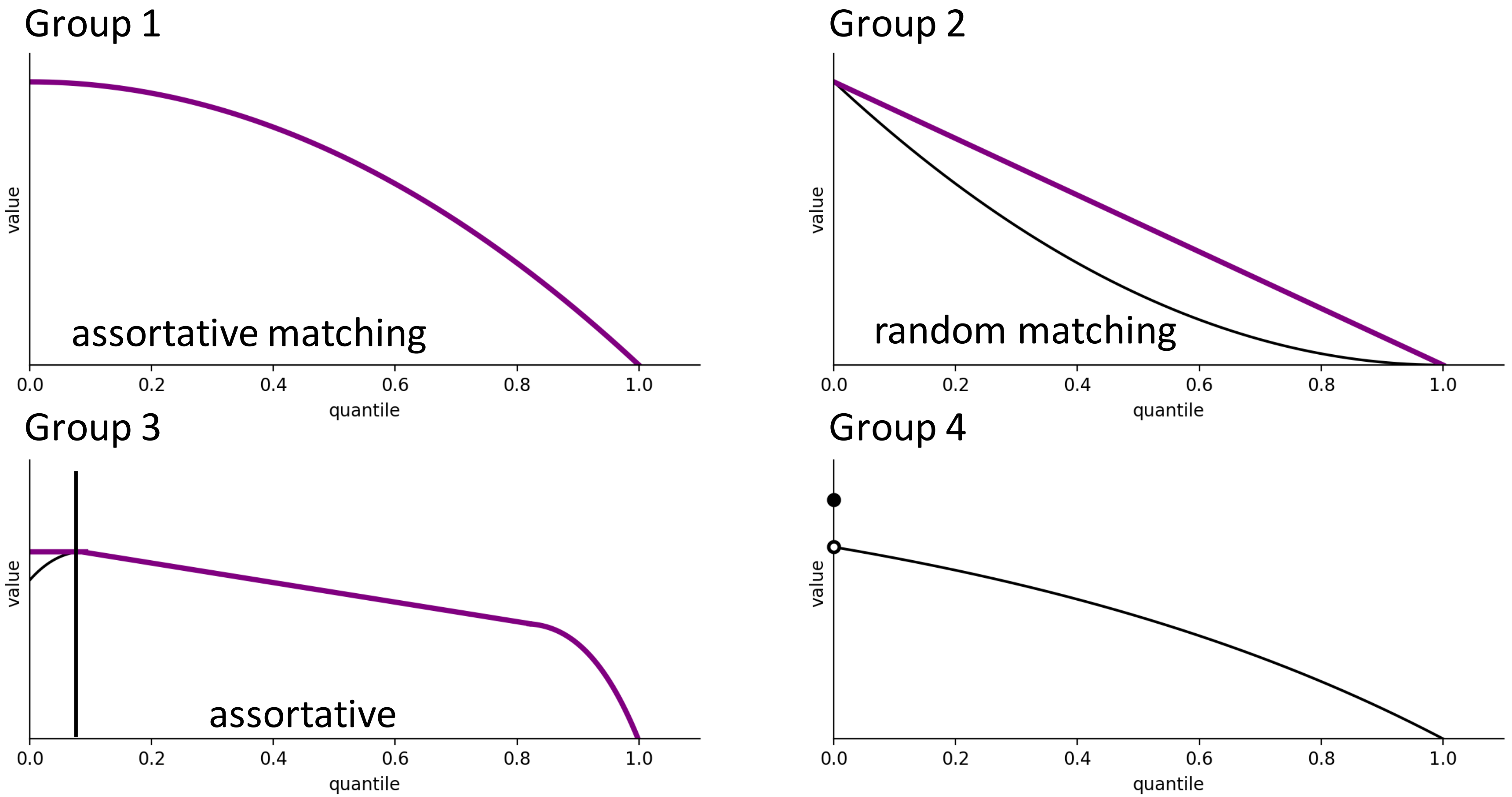
Derivation of Optimal Mechanism: within-group allocation



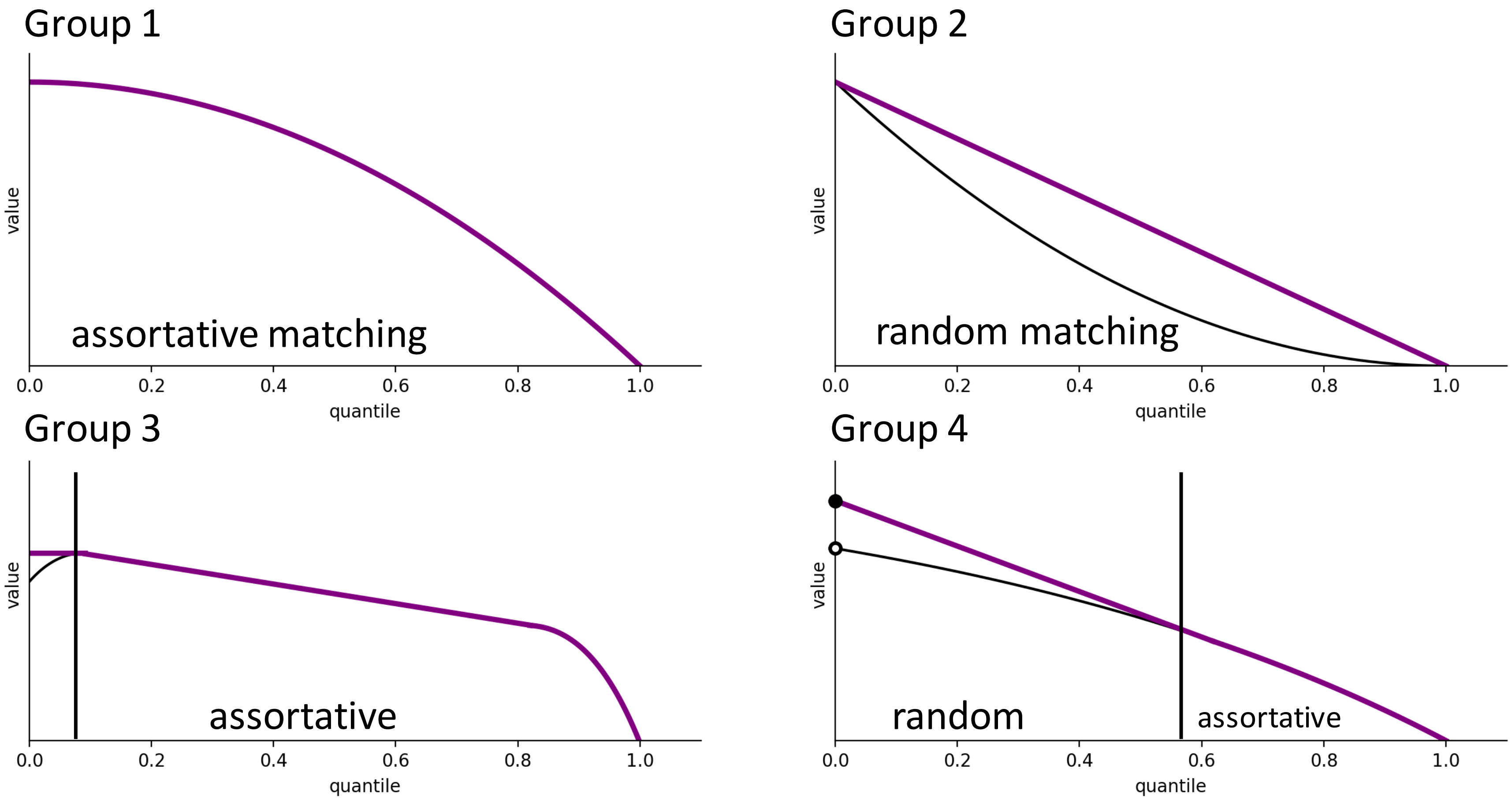
Derivation of Optimal Mechanism: within-group allocation



Derivation of Optimal Mechanism: within-group allocation



Derivation of Optimal Mechanism: within-group allocation



Setup of Across-Group Allocation

The across problem:

$$\max_{(F_i)_{i\in I}} \left\{ \sum_{i\in I} \mu_i \int_0^1 \mathrm{cd}(\Psi_\mathrm{i})\big(F_i(q)\big)\mathrm{dq} \right\}$$

such that $\sum_{i\in I} \mu_i F_i(q) = F(q), \forall q \in Q$.

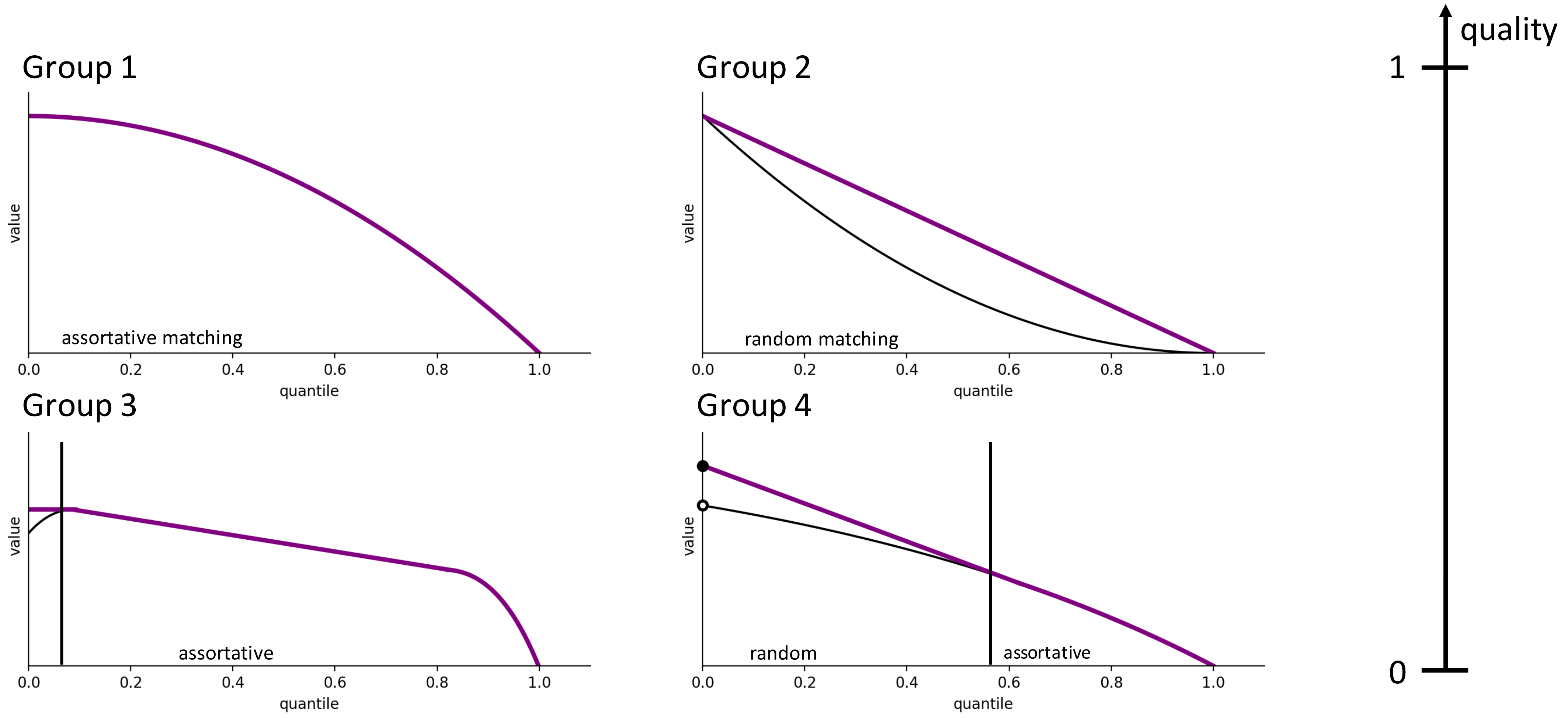
Across-Group Procedure

- Let $s_i(x) \equiv |\mathrm{cd}(\Psi_i)'(x)|$

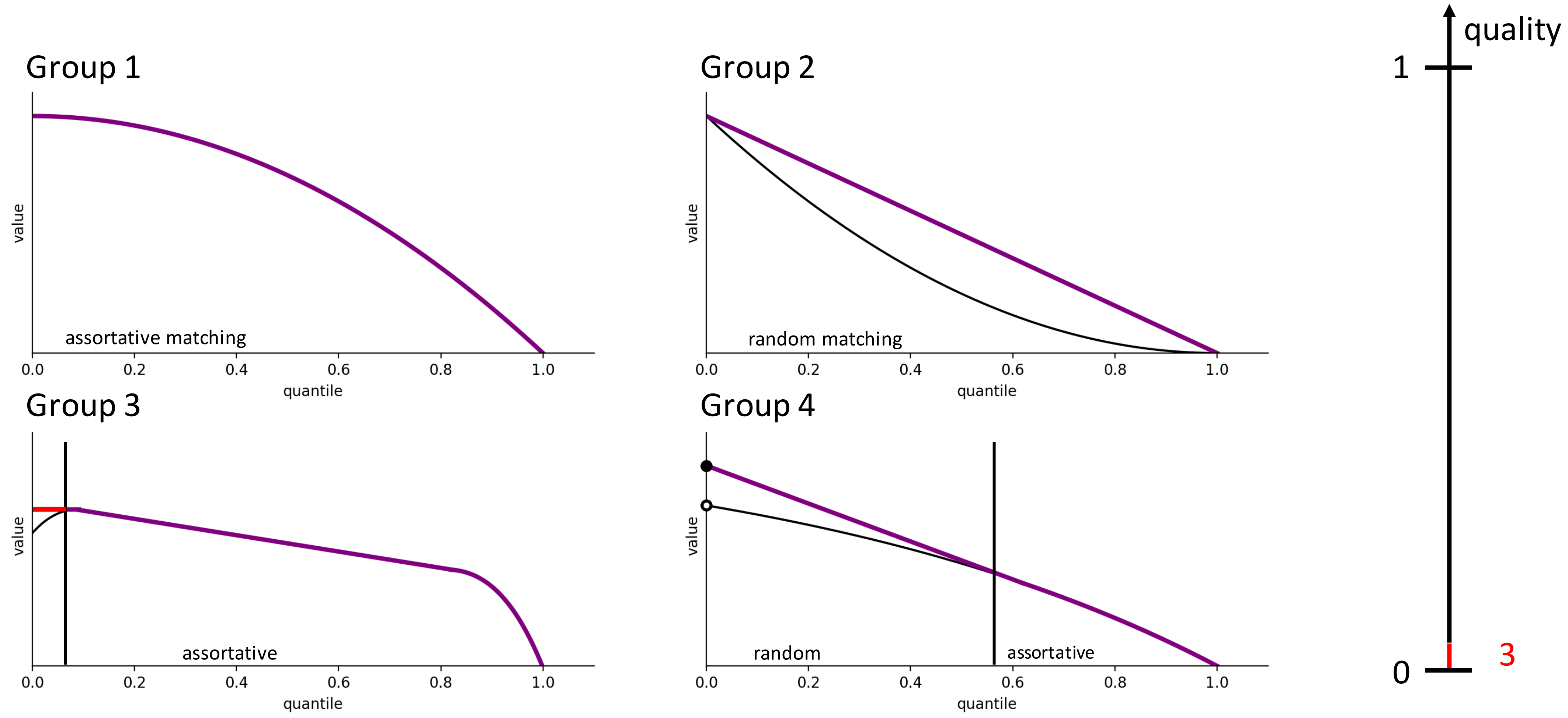
Allocate objects by gradually increasing the CDFs F_i^*, in the order of increasing slopes, keeping track of the running minimum over these slopes.

1. Starting from the lowest quality, increase the CDF F_i^* for group i with the smallest slope s_i at 0.
2. At any q, increase the CDF of group(s) i with the lowest slope s_i at F_i^*.
3. When some F_i^* reaches 1, stop increasing the CDF for that group.

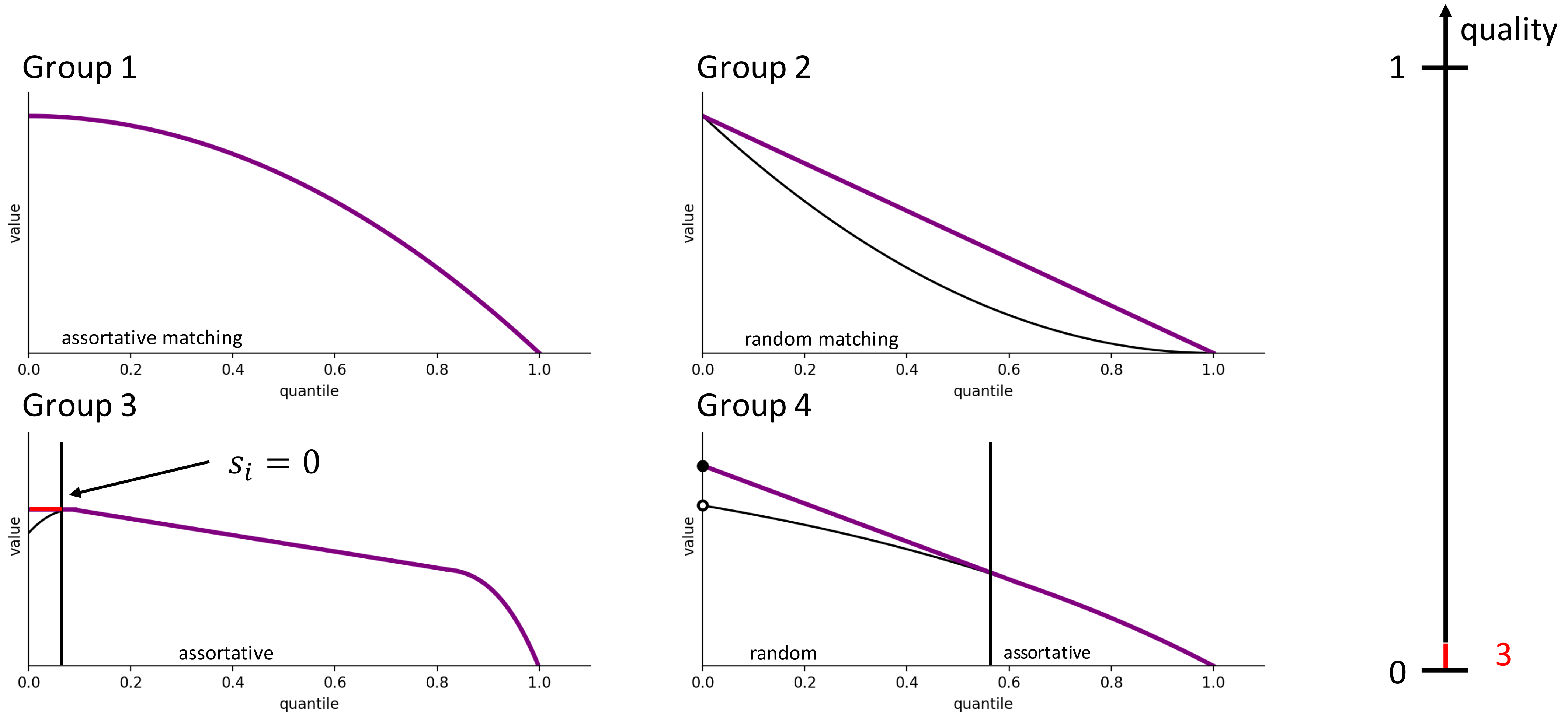
Derivation of Optimal Mechanism: across-group allocation



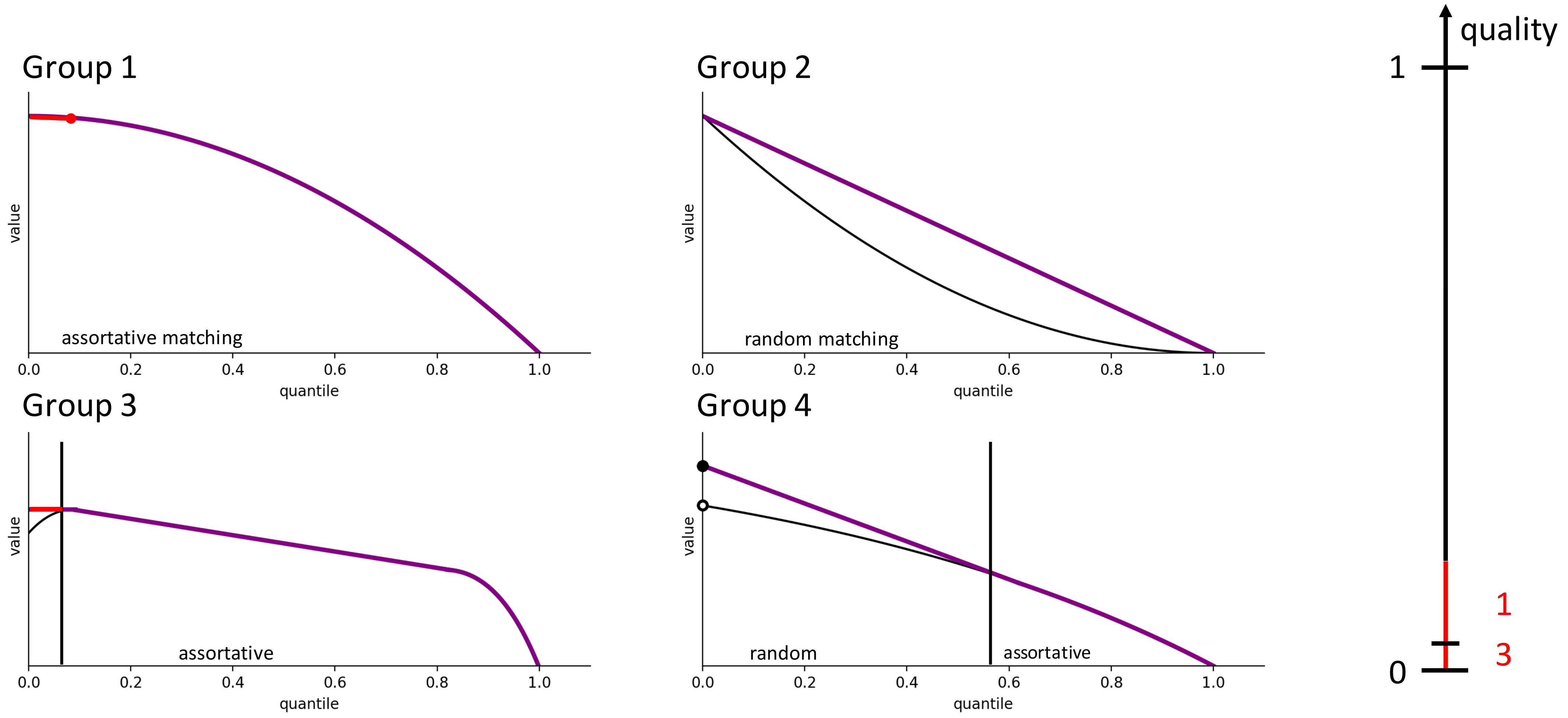
Derivation of Optimal Mechanism: across-group allocation



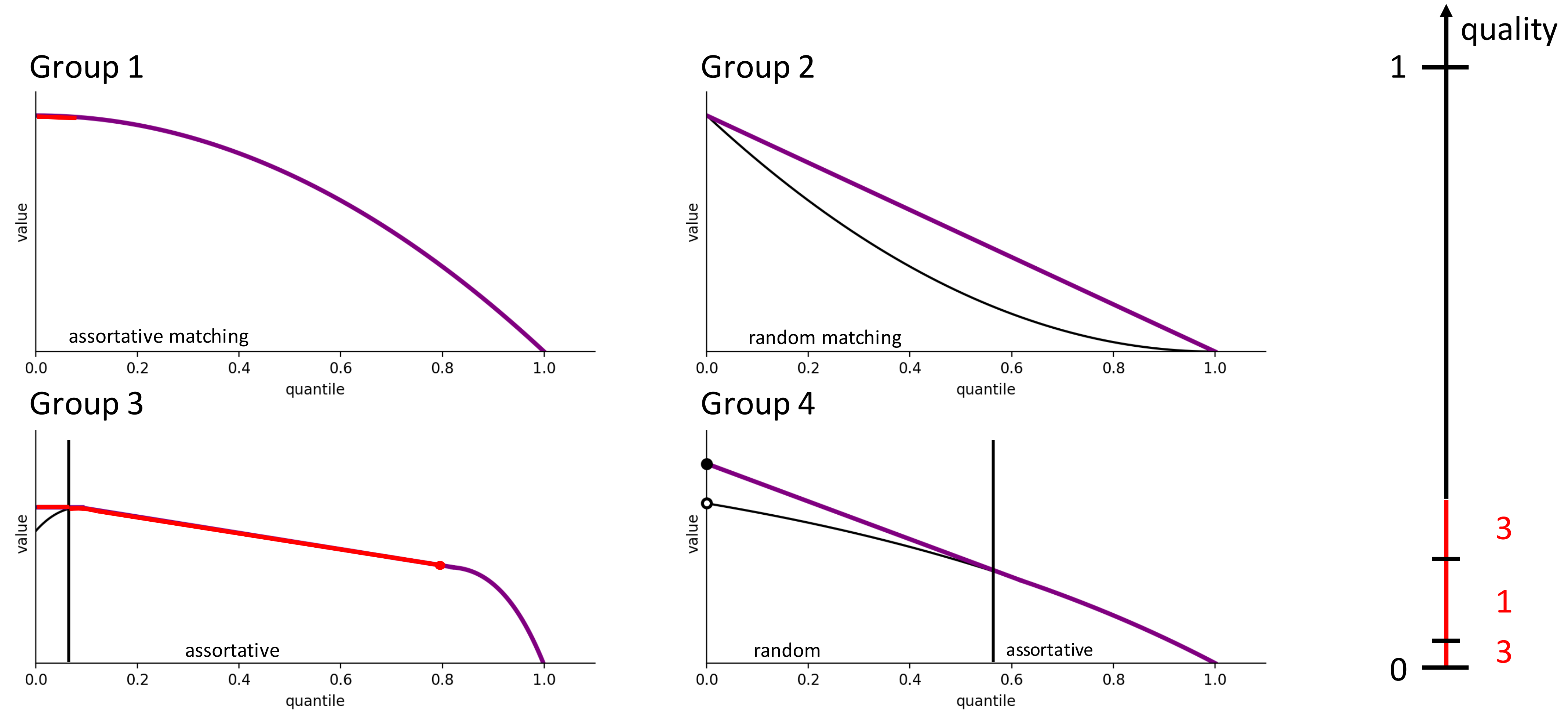
Derivation of Optimal Mechanism: across-group allocation



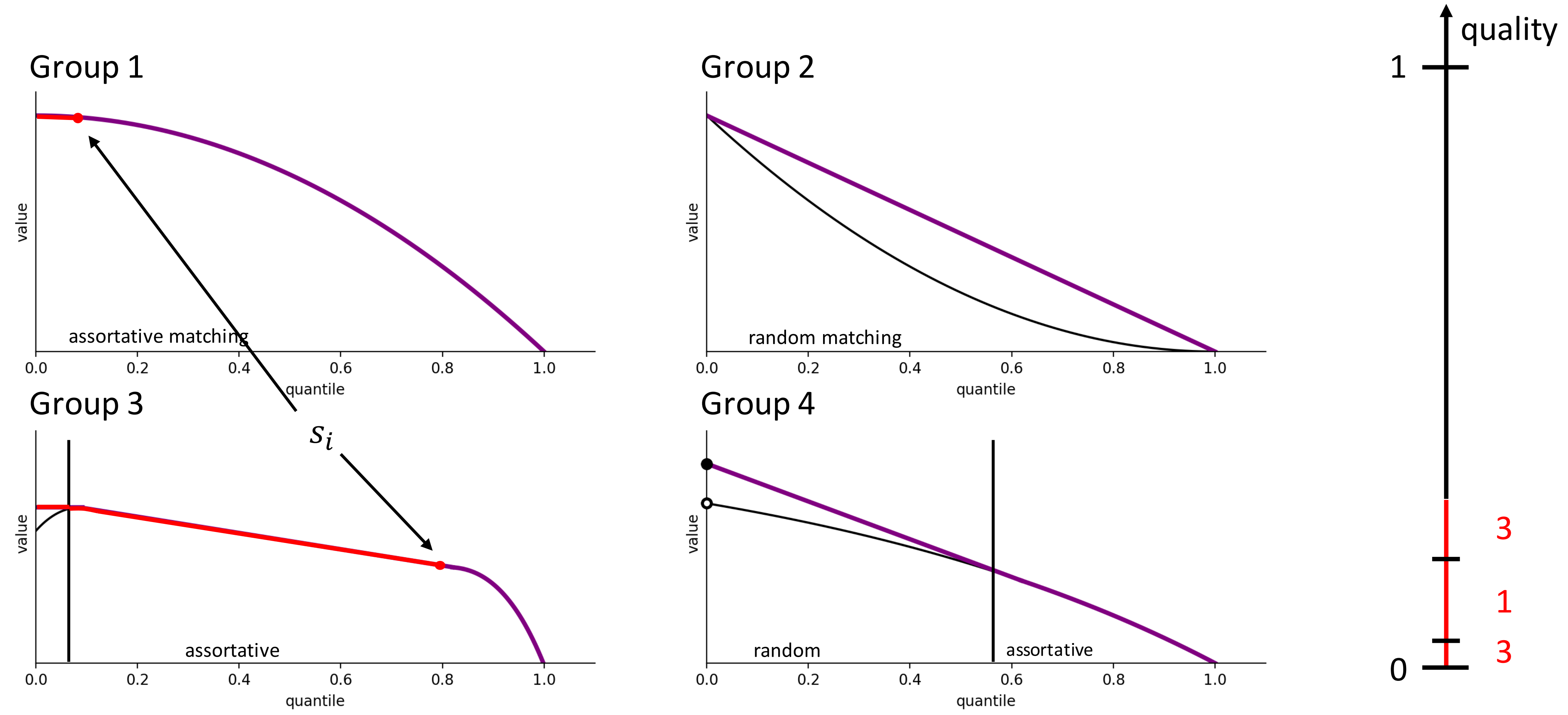
Derivation of Optimal Mechanism: across-group allocation



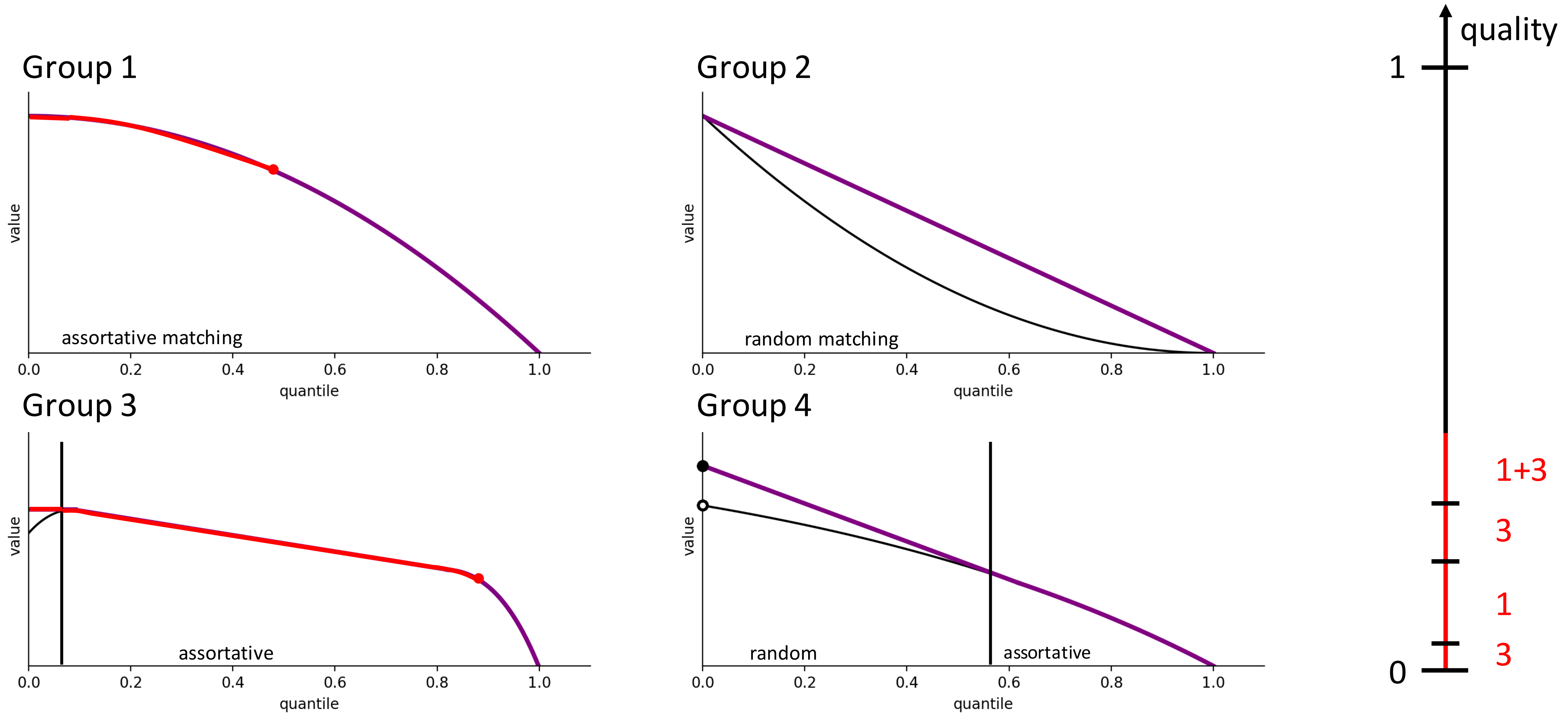
Derivation of Optimal Mechanism: across-group allocation



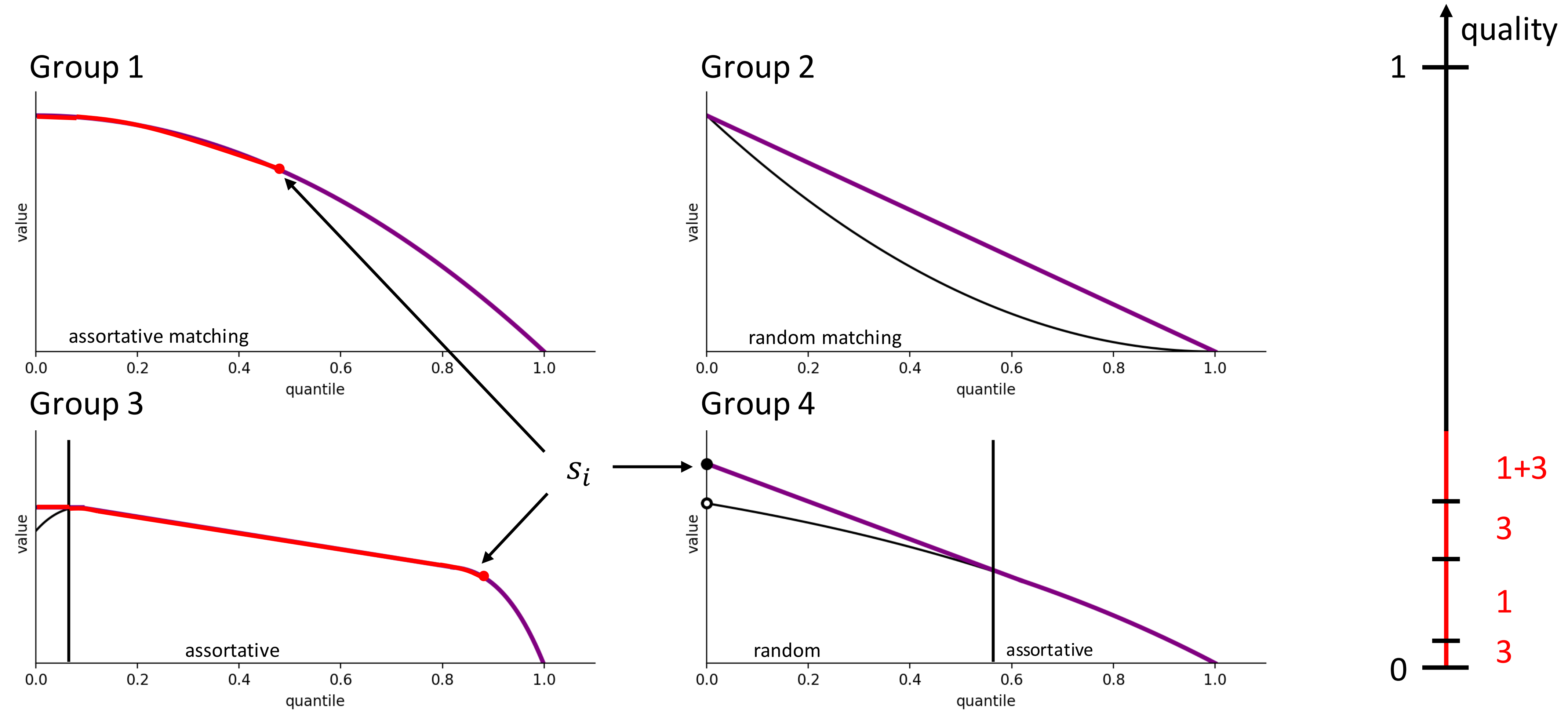
Derivation of Optimal Mechanism: across-group allocation



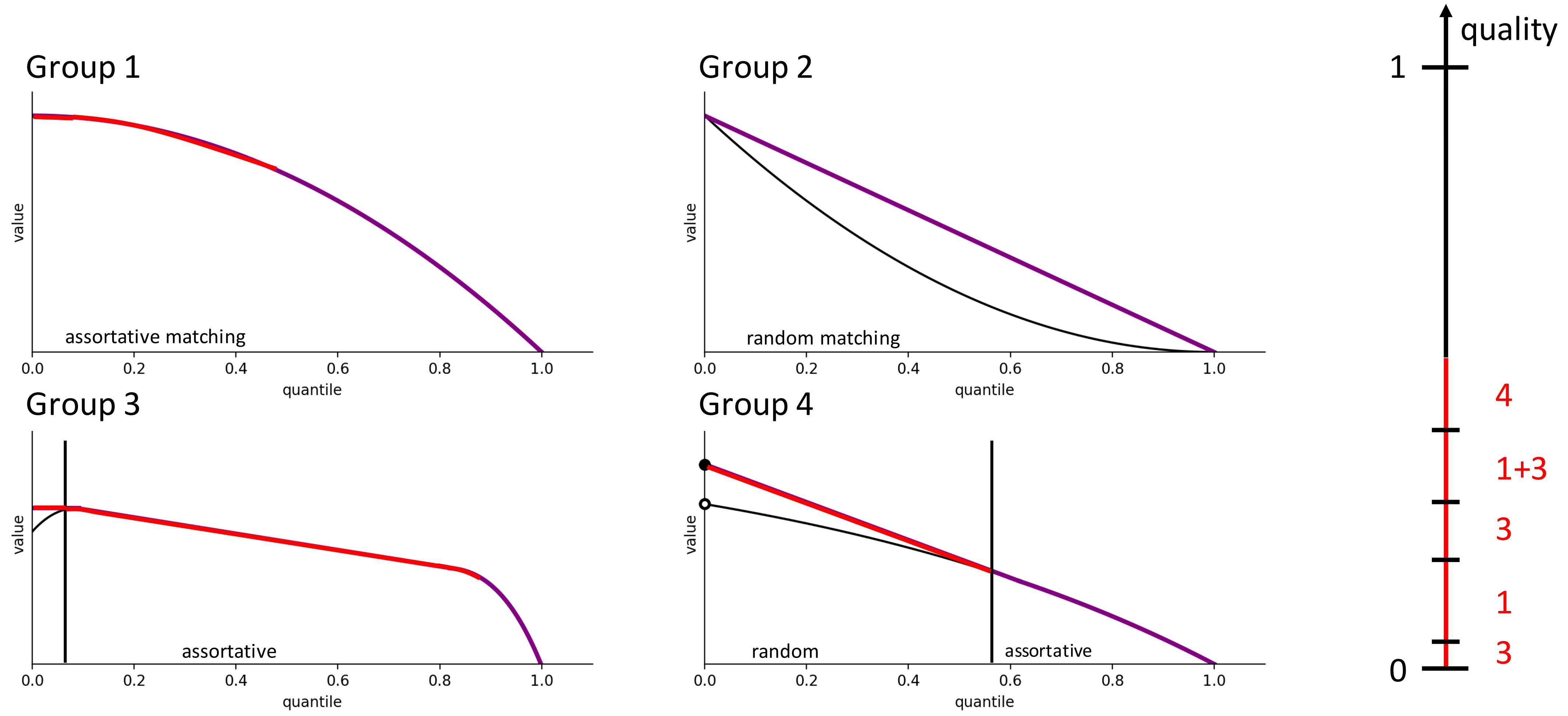
Derivation of Optimal Mechanism: across-group allocation



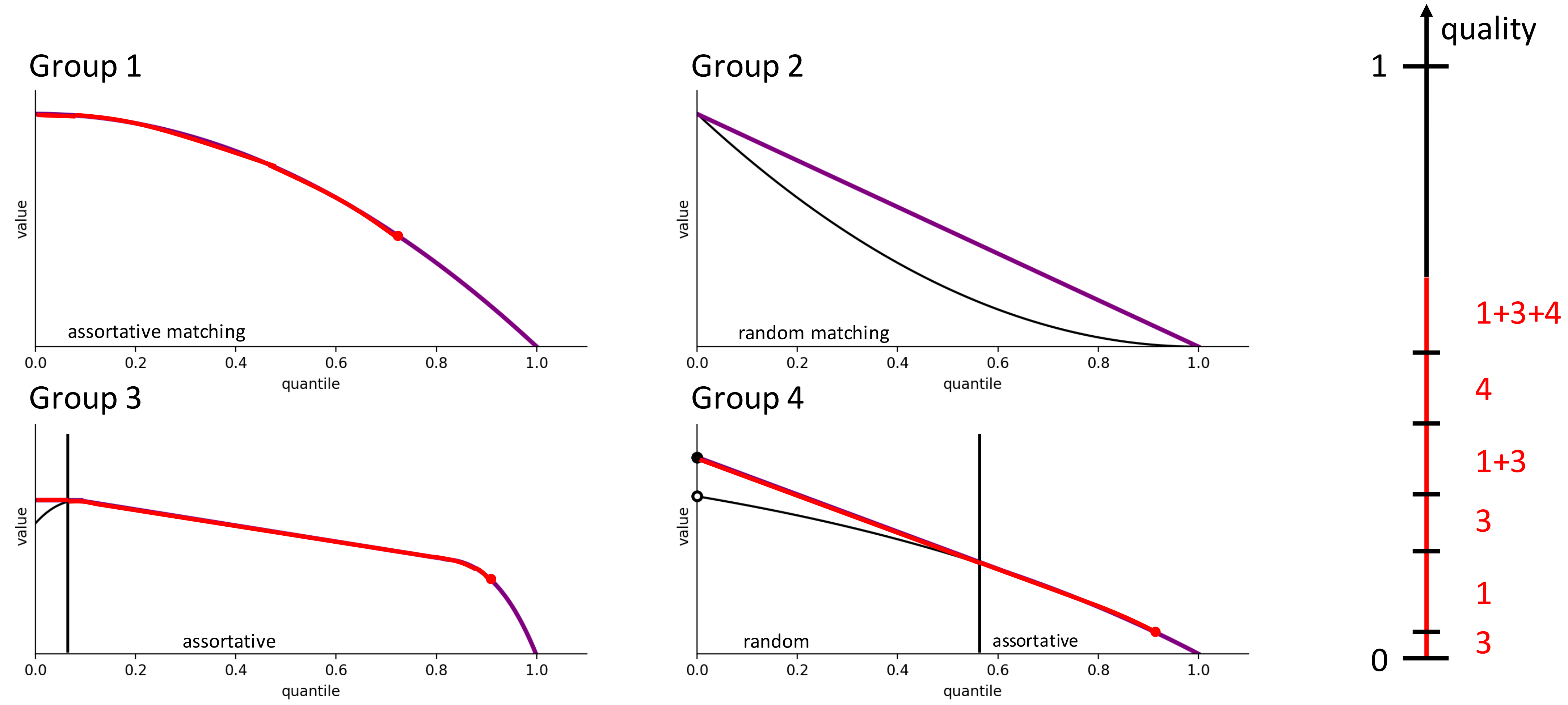
Derivation of Optimal Mechanism: across-group allocation



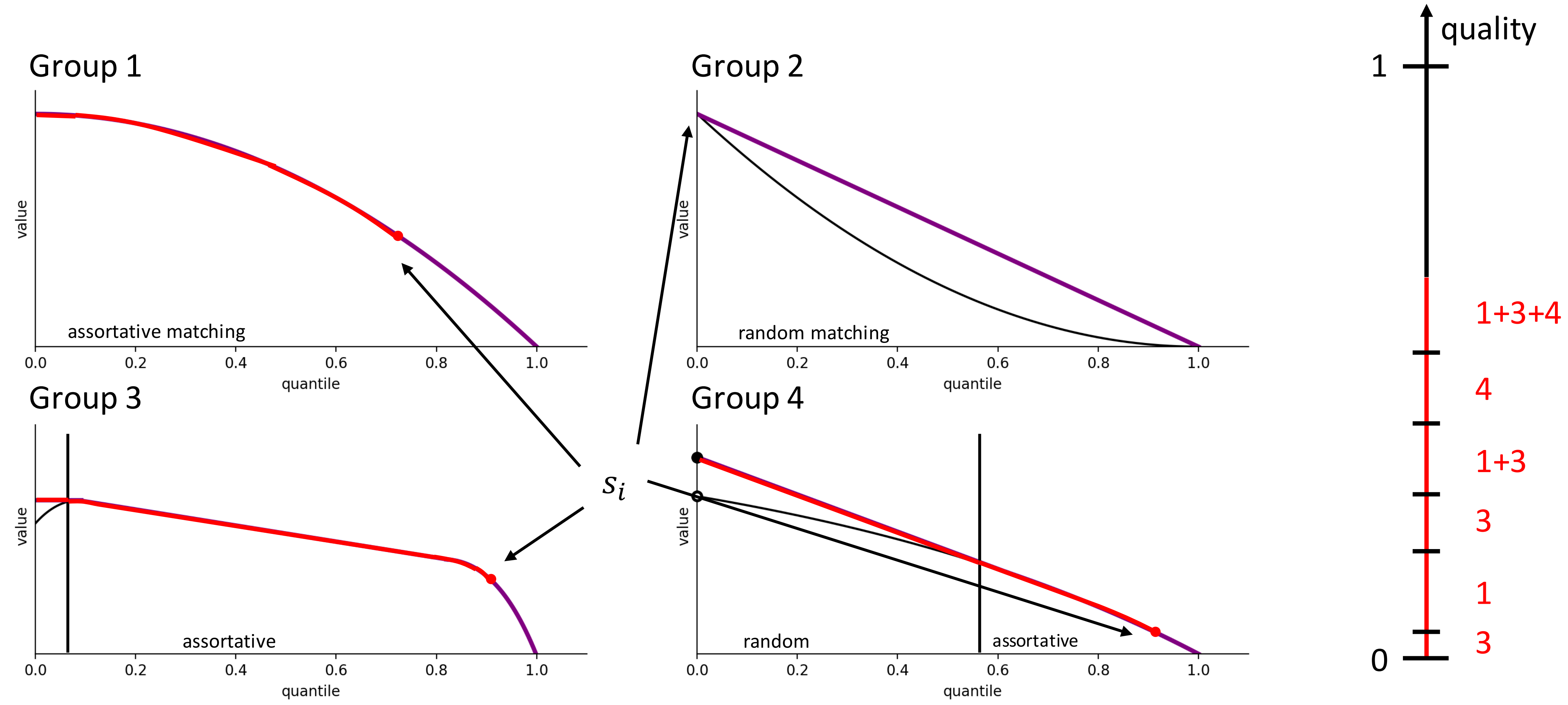
Derivation of Optimal Mechanism: across-group allocation



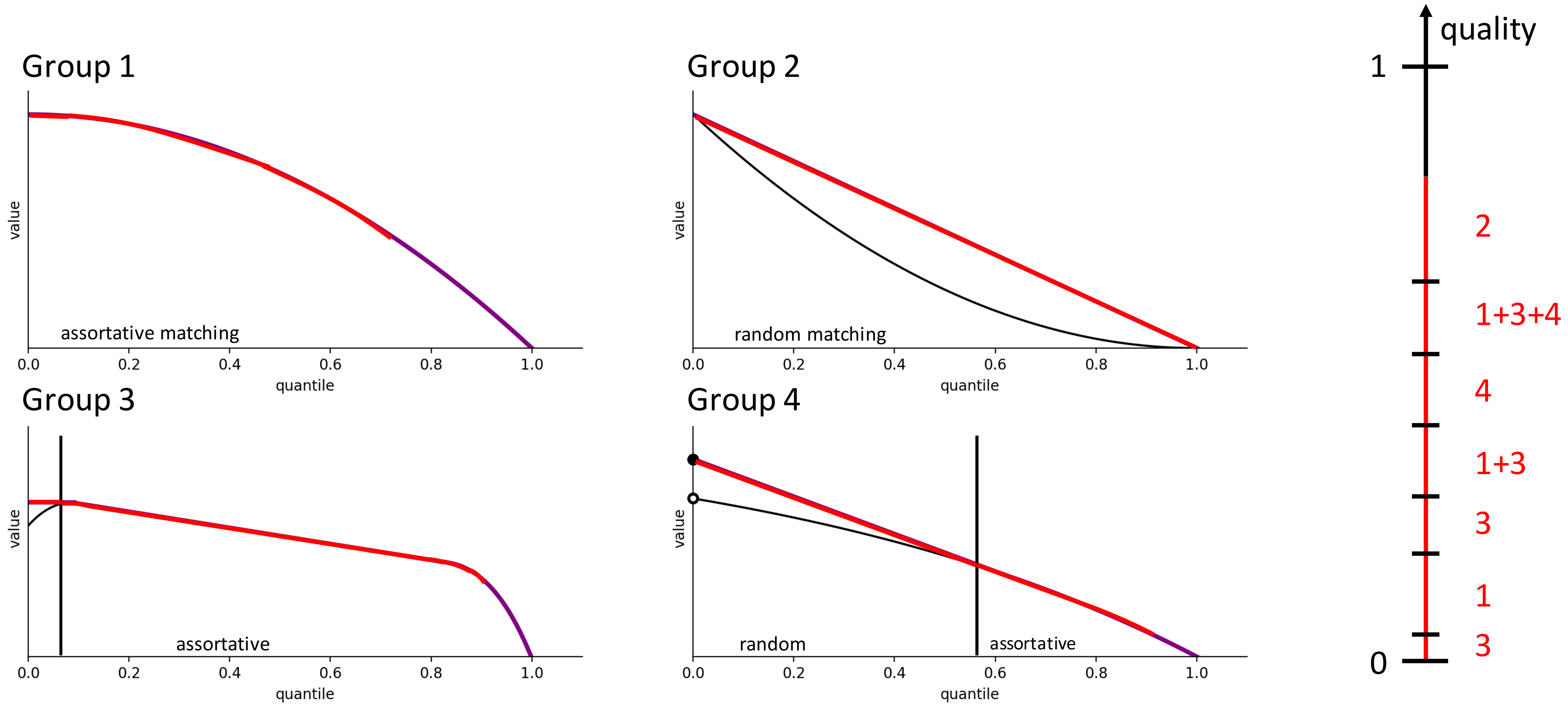
Derivation of Optimal Mechanism: across-group allocation



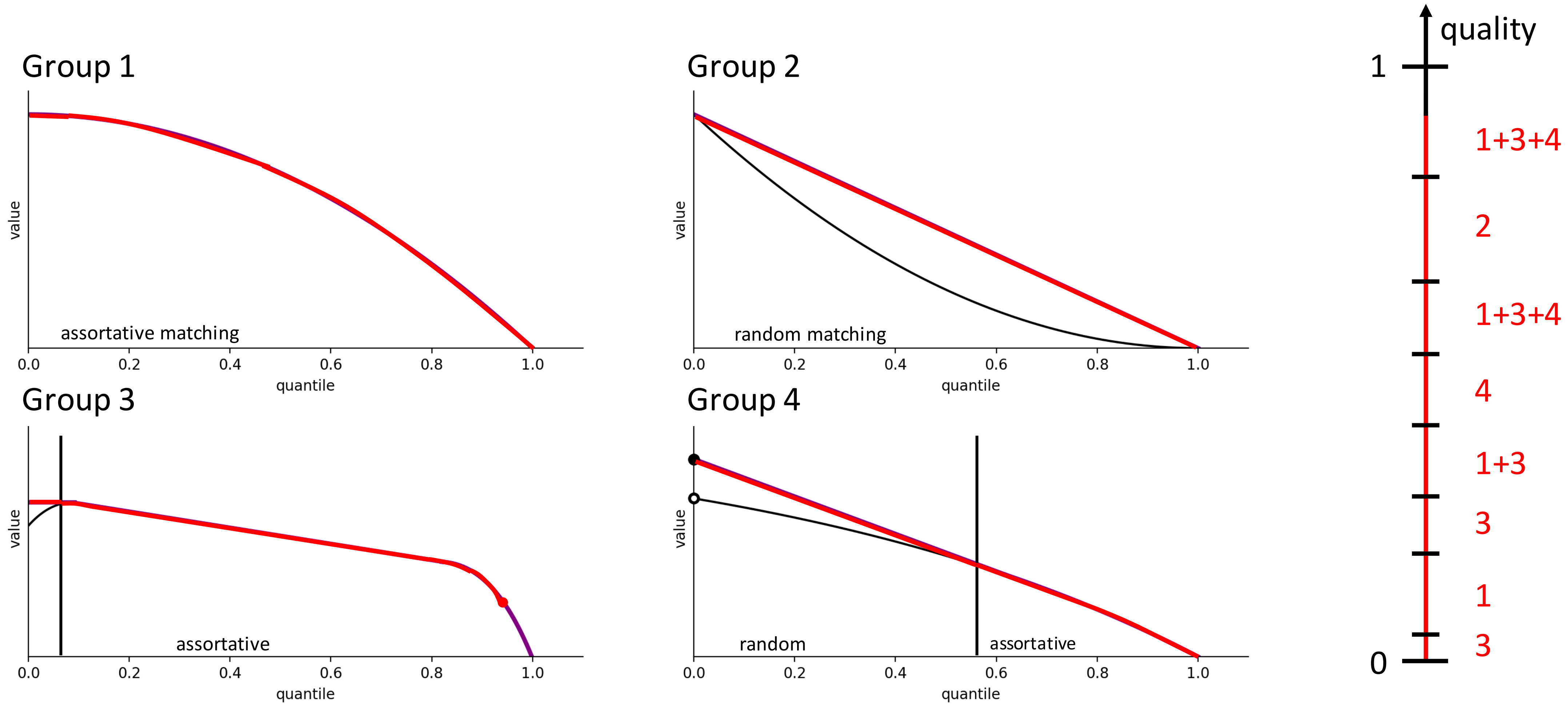
Derivation of Optimal Mechanism: across-group allocation



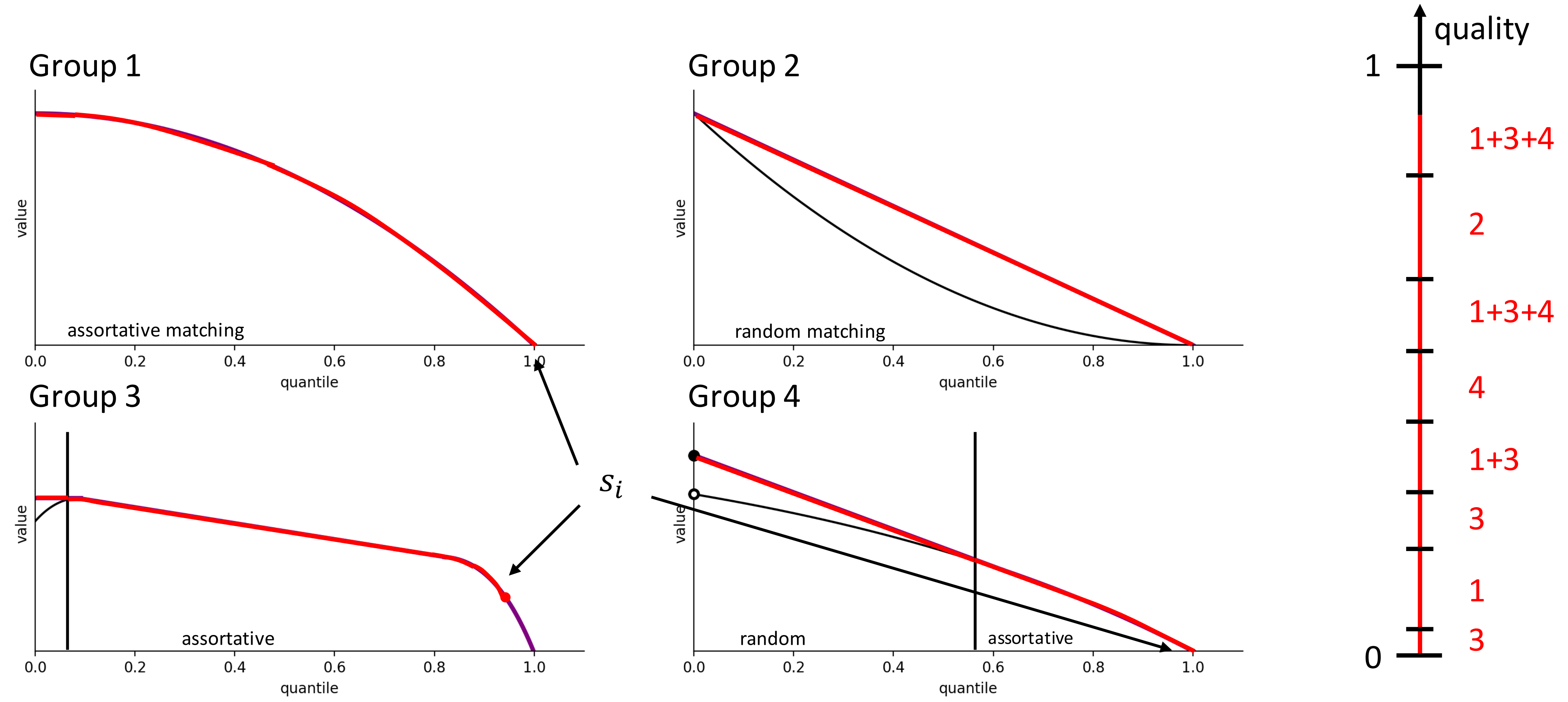
Derivation of Optimal Mechanism: across-group allocation



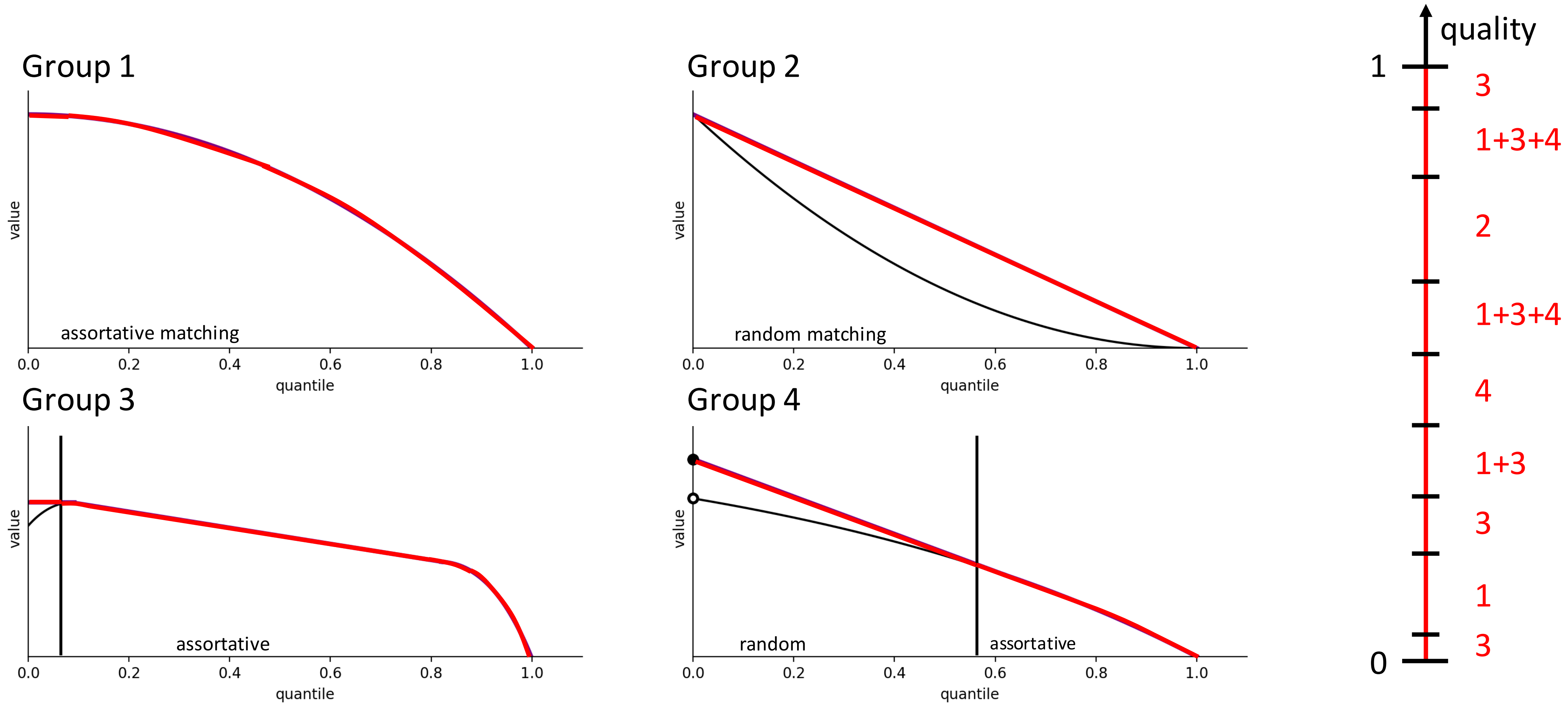
Derivation of Optimal Mechanism: across-group allocation



Derivation of Optimal Mechanism: across-group allocation



Derivation of Optimal Mechanism: across-group allocation



When to use in-kind redistribution?

- Roughly: When observed variables (i) uncover inequality in the unobserved welfare weights.

1. **Label-revealed inequality:** When the average Pareto weight on group i exceeds the weight on revenue (i.e., $\overline{\lambda_i} > \alpha$), it is optimal to use at least some in-kind redistribution for **universally desirable goods** (special case: **essential goods**)

   e.g., food stamps (fully random allocation to eligible groups)

When to use in-kind redistribution?

- Roughly: When observed variables (i) uncover inequality in the unobserved welfare weights.

1. **Label-revealed inequality:** When the average Pareto weight on group i exceeds the weight on revenue (i.e., $\overline{\lambda_i} > \alpha$), it is optimal to use at least some in-kind redistribution for **universally desirable goods** (special case: **essential goods**)

   e.g., food stamps (fully random allocation to eligible groups)

2. **WTP-revealed inequality:** When the welfare weights are strongly and negatively correlated with willingness to pay

   e.g., health care (general population)

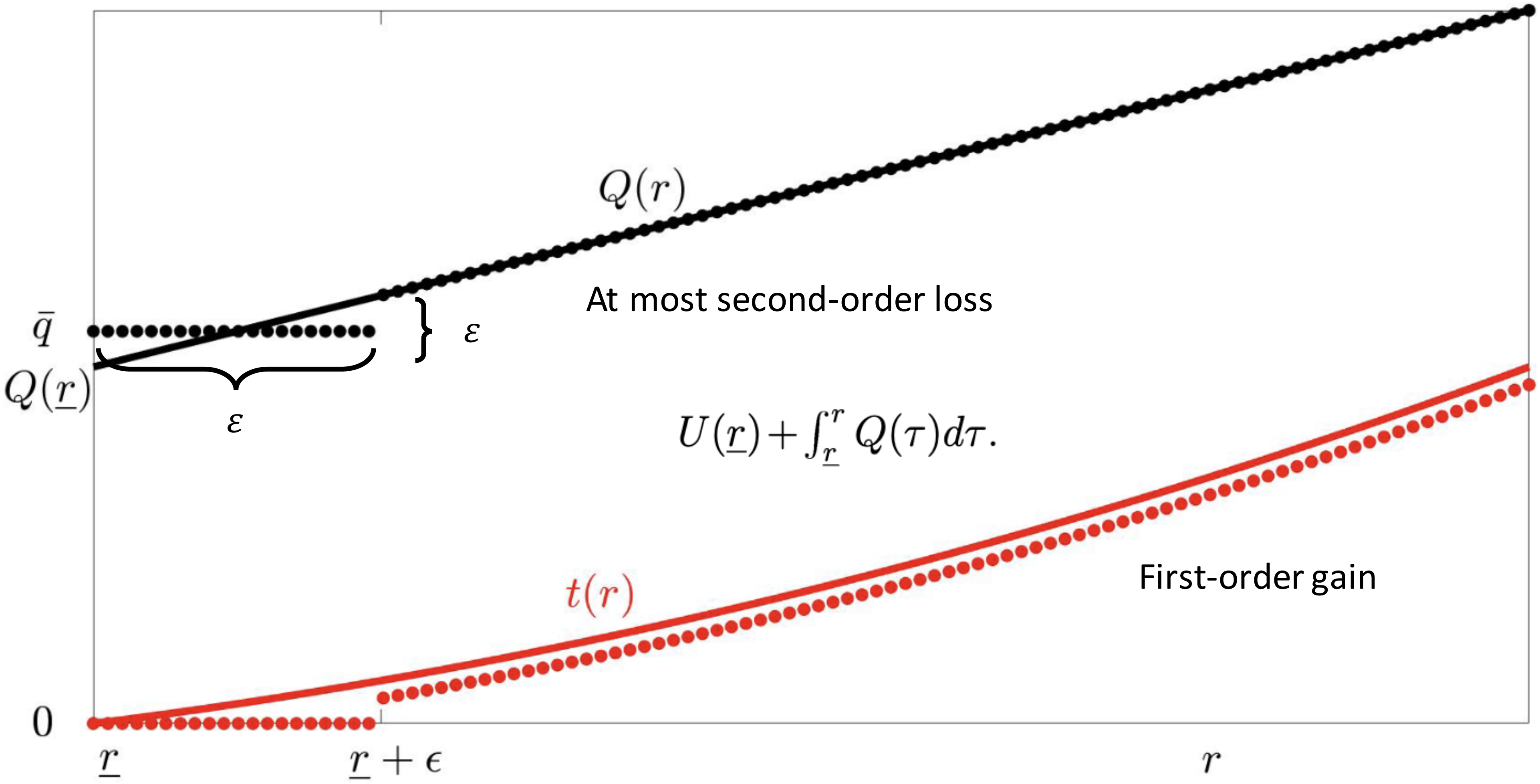
Label-Revealed Inequality

Proposition: If the average Pareto weight $\overline{\lambda}_i$ for group i is strictly larger than the weight on revenue α, and the good is universally desirable $\left(\underline{r_i} > 0\right)$, then there exists $r_i^* > \underline{r_i}$ such that the allocation is random at a price of 0 for all types $r \leq r_i^*$.

"Wrong" intuition: A random allocation for free increases the welfare of agents with lowest willingness to pay

Correct Intuition: A random allocation for free enables the designer to lower prices for all agents

Label-Revealed Inequality (Intuition)



$$[U^{\epsilon}(\underline{r}) + \int_{\underline{r}}^{r} Q^{\epsilon}(\tau)d\tau] - [U(\underline{r}) + \int_{\underline{r}}^{r} Q(\tau)d\tau] = (q^{\epsilon} - Q(\underline{r}))\underline{r} + \int_{\underline{r}}^{\min\{r,\underline{r}+\epsilon\}} (q^{\epsilon} - Q(\tau))d\tau$$

WTP-Revealed Inequality

Proposition: Suppose that $\alpha \geq \overline{\lambda}_i$ and $r_i = 0$. Then, every optimal mechanism provides a random allocation to agents with willingness to pay in some (non-degenerate) interval if and only if $V_i(r) = \alpha J_i(r) + \Lambda_i(r) h_i(r)$ is not non-decreasing.

Recall:

$$h_i(r) \equiv \frac{1 - G_i(r)}{g_i(r)}, \qquad J_i(r) \equiv r - \frac{1 - G_i(r)}{g_i(r)},$$

$$\text{and } \Lambda_i(r) \equiv \mathbb{E}_{\tilde{r} \sim G_i}\left[\lambda_i(\tilde{r}) | \tilde{r} \geq r\right]$$

WTP-Revealed Inequality

Proposition: Suppose that $\alpha \geq \overline{\lambda}_i$ and $r_i = 0$. Then, every optimal mechanism provides a random allocation to agents with willingness to pay in some (non-degenerate) interval if and only if $V_i(r) = \alpha J_i(r) + \Lambda_i(r) h_i(r)$ is not non-decreasing.

Economic intuition: Random allocation will be used for a subset of agents if for some r, we have:

$$\alpha + \Lambda'_i(r) h_i(r) + (\Lambda_i(r) - \alpha) h'_i(r) < 0$$

When to use market mechanisms?

Roughly: when the revenue and efficiency motives dominate vs. redistributive concerns.

1. **Revenue maximization**: When the weight on revenue (α) is above the average Pareto weight $(\overline{\lambda_i})$, some assortative matching is optimal:

   e.g., allocating goods to corporations (oil leases, spectrum licenses)

   e.g., when lump-sum transfers to the target population are feasible

When to use market mechanisms?

Roughly: when the revenue and efficiency motives dominate vs. redistributive concerns.

1. **Revenue maximization:** When the weight on revenue (α) is above the average Pareto weight $(\overline{\lambda_i})$, some assortative matching is optimal:

   e.g., allocating goods to corporations (oil leases, spectrum licenses)

   e.g., when lump-sum transfers to the target population are feasible

2. **Efficiency maximization:** When welfare weights are not strongly correlated with WTP

   e.g., small dispersion in welfare weights to begin with

   e.g., large dispersion in welfare weights but little correlation with WTP

Assortative Matching at the Top

Proposition: If Pareto weights are non-increasing, and $\alpha \geq \overline{\lambda_i}$, then there is assortative matching at the top, i.e., for all types $r \geq r_i^*$ for some $r_i^* \geq \underline{r_i}$.

The assumptions guarantee that the revenue-maximizing motive dominates for high enough types.

The assumption $\alpha \geq \overline{\lambda}_i$ holds when lump-sum transfers are allowed.

Even if there is random allocation "at the bottom," it might be a good idea to have a price gradient "at the top."

In-Kind Transfer of Intermediate Quality

Proposition: Suppose that there are two groups and that in group 1 there is assortative matching (of allocated objects), and in group 2 there is fully random matching.

Then, there exist $\underline{q} \leq \overline{q}$ such that group 2 gets objects of quality $[\underline{q}, \overline{q}]$, and group 1 gets objects of quality $[0, \underline{q}) \cup (\overline{q}, 1]$

In-Kind Transfer of Intermediate Quality

Proposition: Suppose that there are two groups and that in group 1 there is assortative matching (of allocated objects), and in group 2 there is fully random matching.

Then, there exist $\underline{q} \leq \overline{q}$ such that group 2 gets objects of quality $[\underline{q}, \overline{q}]$, and group 1 gets objects of quality $[0, \underline{q}) \cup (\overline{q}, 1]$.

Intuition: Under random matching, only expected quality matters.

But under assortative matching, it is key to have dispersion in quality (easiest to see for revenue maximization—the allocation of lower types is reduced to minimize the information rents of higher types)

Concluding Remarks

Analysis uncovers the importance of three factors in optimal object allocation under general redistributive concerns:

1. Correlation between the unobserved welfare weights and the information that the designer can elicit or observe directly;
2. The role of raising revenue and availability of lump-sum transfers;
3. Whether the good is essential or not.