**Inequality & Ordeals Reading Group** — **HR '08**
**Fall 2023** — Kira Goldner

# Single-Parameter Optimal Utility Maximization

The following is based on [1].

## Bayesian Stages and Interim Rules

*interim*: Values $v_i$ have been drawn; $i$ only knows their own valuation, and thus the updated prior $\mathbf{F} \mid_{v_i}$.

**Definition 1.** We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given $i$'s valuation:

$$x_i(v_i) = \Pr_{\mathbf{F}}[x_i(\mathbf{v}) = 1 \mid v_i] = \mathbb{E}_{\mathbf{F}}[x_i(\mathbf{v}) \mid v_i] \qquad \text{and} \qquad p_i(v_i) = \mathbb{E}_{\mathbf{F}}[p_i(\mathbf{v}) \mid v_i].$$

Our definition of Bayesian Incentive-Compatibility then follows:

**Definition 2.** A mechanism with *interim* allocation rule $x$ and *interim* payment rule $p$ is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

Using these, we can more easily prove the BIC/BNE versions of Myerson's Lemma and the Revelation Principle.

## Maximizing Expected Revenue via Virtual Welfare

Recall that the revelation principle says that it's without loss to focus only on truthful mechanisms.

Payment is determined by the allocation:

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz$$

We want to maximize $\mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})]$.

We derive that $\mathbb{E}_{v_i \sim F_i}[p_i(v_i, \mathbf{v}_{-i})] = \mathbb{E}_{v_i\sim F_i}[\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})]$ where

$$\varphi_i(v_i) = v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}$$

is the Myersonian virtual value. Then

$$\textsc{Revenue} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i \varphi_i(v_i) x_i(\mathbf{v})] = \textsc{Virtual Welfare}$$

by taking $\mathbb{E}_{\mathbf{v}_{-i}\sim\mathbf{F}_{-i}}$ of both sides of our previous equation.

## Maximizing Utility via Virtual Welfare

To maximize $\text{UTILITY} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i x_i(\mathbf{v})v_i - p_i(\mathbf{v})]$ we can then substitute in our above virtual welfare for the revenue term:

$$\begin{aligned}
\text{UTILITY} &= \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i x_i(\mathbf{v})v_i - p_i(\mathbf{v})] \\
&= E_{\mathbf{v}\sim\mathbf{F}}[\sum_i x_i(\mathbf{v})v_i] - \text{REVENUE} \\
&= E_{\mathbf{v}\sim\mathbf{F}}[\sum_i x_i(\mathbf{v})v_i] - \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i \varphi_i(v_i)x_i(\mathbf{v})] \\
&= \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i x_i(\mathbf{v}) \left( v_i - v_i + \frac{[1 - F_i(v_i)]}{f_i(v_i)} \right)] \\
&= \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i x_i(\mathbf{v})\theta_i(v_i)]
\end{aligned}$$

where

$$\theta_i(v_i) = \frac{[1 - F_i(v_i)]}{f_i(v_i)}$$

Given this conclusion, how should we design our allocation rule $x$ to maximize expected virtual welfare (expected revenue)? Give the item to the bidder with the highest *virtual* value $\theta$!

When would this cause a problem with incentive-compatibility? When the corresponding $x$ isn't monotone! When is this monotone? For *anti-MHR* distributions.

## Ironing

- Convert virtual values to quantile space: $h(q) = \theta(q) := \theta(F^{-1}(q))$ — the $v$ corresponding to that $q$.
- Integrate to get the curve: $H(q) = \int_0^q h(r)\,dr$.
- Take the convex closure: define $\bar{H}$ as the largest convex function bounded above by $H$ for all $q \in [0, 1]$.
- Take the derivative: $\bar{h}(q)$ is the derivate of $H(q)$ extended to all of $[0, 1]$ by right-continuity.
- Convert back to value space: $\bar{\theta}(v) = \bar{h}(F(v))$.

**Claim 1.**

$$\mathbb{E}_{\mathbf{v}}[\sum_i x_i(\mathbf{v})\theta_i(v_i)] \le \mathbb{E}_{\mathbf{v}}[\sum_i x_i(\mathbf{v})\bar{\theta}_i(v_i)]$$

with equality if and only if $x_i'(v) = 0$ whenever $\bar{H}(F(v)) < H(F(v))$

**Claim 2.** The mechanisms that maximize utility are precisely those that

(1) maximize virtual ($\theta$) welfare for every input $\mathbf{v}$ and

(2) the allocation is non-increasing (its derivative is zero) for every bidder whenever the utility curve is ironed.
(In math: for all $i$, $x_i'(v) = 0$ whenever $\bar{H}(F(v)) < H(F(v))$.)

Optimal mechanisms:

- Anti-MHR: Vickrey.
- MHR: Lottery.
- In-between: For iid, Vickrey with tie-buckets.

**Definition 3.** A $k$-unit $(p, q)$-lottery denoted $\text{Lot}_{p,q}$ allocates as follows for $p > q$. Let the agents that bid over $p$ be the "priority" agents.

**a.** If there are at most $k$ agents with bid over $q$ (including the priority agents), allocate to them all at a price of $q$.

**b.** If there are $k$ or more priority agents, allocate to them at a price of $p$, running a lottery to determine who gets allocated.

**c.** Otherwise, allocate to the priority agents with certainty at price $\frac{k-s+1}{t+s}q + \frac{s+t-k}{t+1}p$ and lottery the remaining $s$ agents above $q$ at a price of $q$.

## References

[1] Jason D. Hartline and Tim Roughgarden. Optimal mechanism design and money burning. In *STOC '08: Proceedings of the 40th annual ACM symposium on Theory of computing*, pages 75–84, New York, NY, USA, 2008. ACM.